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Dimensions and Bases of Hierarchical Tensor-Product Spline Spaces

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Dimensions and Bases of Hierarchical Tensor-Product Spline Spaces

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Abstract

Given a grid in $\mathbb{R}^d$, consisting of d bi-infinite sequences of hyperplanes (possibly with multiplicities) which are orthogonal to the d axes of the coordinate system, we consider the spaces of tensor-product spline functions of a given degree on a multi-cell domain. Such a domain consists of finitely many of the cells which are defined by the grid. A piecewise polynomial function belongs to the spline space if its polynomial pieces on adjacent cells have a contact according to the multiplicity of the hyperplanes in the grid. We prove that the connected components of the associated set of tensor-product B-splines whose support intersects the multi-cell domain forms a basis of this spline space. More precisely, if the intersection of the support of a tensor-product B-spline with the multi-cell domain consists of several connected components, then each of them contributes one basis function.

In the second part of the paper we consider hierarchical spline spaces, which are defined by specifying a hierarchy of spline spaces on multi-cell domains (as analyzed in the first part) and an associated domain hierarchy. By adapting the techniques from [12] to this more general setting we are able to derive a basis of the hierarchical spline space provided that the domain hierarchy satisfies certain mild assumptions.

1 Introduction

Hierarchical tensor-product splines were introduced by Forsey and Bartels [11] as a tool for adaptive surface modeling. About ten years later, Kraft [17] defined a basis and a quasi-interpolation operator for this spline spaces. At the same time, these splines were used for adaptive surface fitting [15].

Since the advent of isogeometric analysis (IGA), which was established in 2005 as a new approach to bridge the gap between analysis and design in engineering applications [6], there is an renewed interest in adaptive and hierarchical techniques for tensor-product splines.

While the early approaches to adaptive refinement in IGA were based mostly on T-splines [1, 10] which originated more recently than hierachical B-splines in geometric modeling [26], it was soon observed that hierarchical B-splines possess a number of useful theoretical and practical properties which make them well suited for numerical simulation based on IGA [27]. It was shown that these adaptive splines can be equipped with a simple basis which provides the partition of unity and improves the sparsity properties [13], that this basis is strongly stable with respect to the L_∞ norm [14] and that these functions can be implemented efficiently using standard data structures [16]. Meanwhile there is a growing number of papers on hierarchical methods in IGA [5, 18, 25].

Recently, the new approach of locally refined splines has emerged [9]. Currently it is too early to judge whether this will become a valuable alternative to the existing approaches.

Simultaneously, adaptive and locally refined spline spaces were considered from an algebraic viewpoint. More precisely, given a certain partition of the domain into axis aligned boxes, the general goal is to determine the dimension of the spline space (which contains all piecewise polynomial functions of a certain degree and smoothness) and to construct a basis. Several valuable contributions for various cases were described in the rich literature on this topic [7, 8, 19, 20, 21, 22, 23]

Under certain conditions, the hierarchical spline basis spans the entire space of all piecewise polynomial functions of the given degree and smoothness which are defined on the underlying grid (which possesses T-joints) and is therefore *complete*. Such conditions were first studied in [12] for the bivariate case of uniform degrees, dyadic refinement and maximum smoothness. A number of recent manuscripts and preprints presented several generalizations, based on the algebraic framework (homology techniques) described in [22]. We mention the recent preprint [2] which addresses the three-dimensional case and the additional preprints [3, 4].

The present paper introduces a different approach. It is based on the observation that the completeness of the hierarchical spline space can be studied without using advanced results from algebraic homology, employing solely standard methods from the theory of tensor-product spline functions. The simple approach presented in this paper allows to derive sufficient conditions for complete hierarchical spline spaces in any dimension, for any smoothness and for any degree.

The remainder of this paper consists of two sections. First we analyze the dimensions and the bases of a tensor-product spline function on a multi-cell domain. In the second part, by a slight generalization of the techniques from [12], we derive a simple sufficient condition for the completeness of a hierarchical spline space.

2 Splines on multi–cell domains

This section derives a basis for tensor-product splines on multi-cell domains. After presenting some definitions, we will prove that this spline space is spanned by a basis consisting of all connected components of the tensor-product B-splines whose supports intersects the multi-cell domain.

2.1 Tensor–product B–splines

Given a positive integer d which specifies the dimension of the space, we consider the d–dimensional space $\mathbb{R}^d$ with coordinates $\mathbf{x} = (x^{(1)}, \ldots, x^{(d)})$. In addition, we consider d bi–infinite strictly increasing sequences

$$\left(g_j^{(i)}\right)_{j\in\mathbb{Z}}, \quad g_j^{(i)} < g_{j+1}^{(i)}$$

which will be called the *nodes*. Using these sequences of nodes we define a *grid* G consisting of *grid hyperplanes*

$$G_j^{(i)} = \{\mathbf{x} \in \mathbb{R}^d \mid x^{(i)} = g_j^{(i)}\}$$

with associated *multiplicities* $m_j^{(i)}$, which do not need to be the same for all the hyperplanes in the grid.

In addition we choose a *degree* $\mathbf{p} = (p^{(1)}, \dots, p^{(d)})$, where all $p^{(i)}$ are positive integers. We denote with B the set of tensor product B–splines defined on this grid. More precisely, these tensor–product B–splines are products of d univariate B–splines with the variable $x^{(i)}$, which are defined by the bi–infinite knot vectors

$$(\dots, \underbrace{g_{j-1}^{(i)}, \dots, g_{j-1}^{(i)}}_{m_{j-1}^{(i)} \text{times}}, \underbrace{g_{j}^{(i)}, \dots, g_{j}^{(i)}}_{m_{j}^{(i)} \text{times}}, \underbrace{g_{j+1}^{(i)}, \dots, g_{j+1}^{(i)}}_{m_{j+1}^{(i)} \text{times}}, \dots),$$

where each knot appears as often as specified by the multiplicity of the associated hyperplane. These tensor-product B-splines are well–defined if the multiplicities satisfy

$$1 \le m_j^{(i)} \le p^{(i)} - 1. \tag{1}$$

In the sequel we will denote the tensor-product B-splines $\beta \in B$ simply as *B-splines*.

We consider d indices $j_1, \dots, j_d \in \mathbb{Z}$. The closed set

$$\bigtimes_{i=1}^{d} [g_{j_i-1}^{(i)}, g_{j_i}^{(i)}], \tag{2}$$

which is the Cartesian product of d closed intervals between adjacent nodes, is called a *cell* of the grid. The *set of all cells* will be denoted by C and the individual cells will be denoted by $c \in C$.

Consider a cell $c \in C$. We define the set of all B–splines which act on this cell,

$$B_c = \{\beta \in B \mid c \subset \overline{\operatorname{supp} \beta}\}, \tag{3}$$

where the symbol supp denotes the support of a function, i.e.,

$$\operatorname{supp} g = \{\mathbf{x} \in \mathbb{R}^d \mid g(\mathbf{x}) \ne 0\}.$$

In the case of B–splines, where the multiplicity of the knots is less than the degree, this is an *open* set.

Example 1. Figure 1 shows an example of a set B_c. We consider biquadratic B-splines on a uniform grid with all multiplicities equal to 1 in the plane. The cell c is shown in gray. The basis functions which belong to the set B_c are represented by the small circles in the centers of their supports, which coincide with their Greville points.

Consider a *polynomial f of multi–degree* $\mathbf{p}$, i.e., f is a polynomial with the variables $x^{(i)}$, where the degree with respect to $x^{(i)}$ is at most $p^{(i)}$. We denote the linear space of all polynomials of this kind by $\Pi^{\mathbf{p}}(\mathbb{R}^d)$.

When restricting f and $\Pi^{\mathbf{p}}(\mathbb{R}^d)$ to this cell, we obtain

$$\Pi^{\mathbf{p}}(c) = \{f|_c \mid f \in \Pi^{\mathbf{p}}(\mathbb{R}^d)\}.$$

Each restriction $f|_c$ can be expressed as a linear combination of the tensor–product B–splines in B_c,

$$f|_c(\mathbf{x}) = \sum_{\beta \in B_c} \lambda_c^{\beta}(f|_c) \beta|_c(\mathbf{x}), \quad \mathbf{x} \in c, \tag{4}$$

where $\lambda_c^{\beta}(f|_c)$ is the coefficient of $\beta \in B_c$ in the local representation of the polynomial f on the cell c. Note that f is a polynomial defined on $\mathbb{R}^d$, whereas $f|_c$ is defined on c only.

Example 2. Consider again the example of biquadratic splines in Figure 1. Each cell is influenced by nine basis functions from B_c, and each biquadratic polynomial on the cell can be uniquely represented as a linear combination of these nine functions.

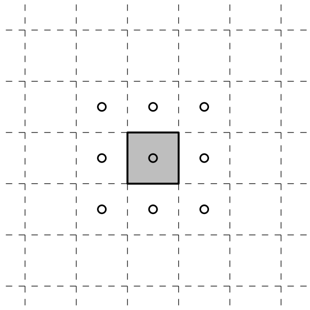


Figure 1: The set of basis functions which act on a single cell.

2.2 Contact of polynomial pieces

We denote the partial derivatives of a polynomial f by

$$\partial_i^j f := \frac{\partial^j f}{\partial (x^{(i)})^j}.$$

Given a polynomial $f|_c$ on a cell c, we define its partial derivatives by considering its canonical extension to $\mathbb{R}^d$,

$$\partial_i^j (f|_c) := (\partial_i^j f)|_c,$$

thereby avoiding the need to consider one–sided limits at the boundary of c.

Consider two cells $c, d \in C$. There exist indices $j_i, k_i \in \mathbf{Z}$ so that

$$c = \bigtimes_{i=1}^{d} [g_{j_i}^{(i)}, g_{j_i+1}^{(i)}] \quad \text{and} \quad d = \bigtimes_{i=1}^{d} [g_{k_i}^{(i)}, g_{k_i+1}^{(i)}].$$

Their intersection is an axis-aligned box whose dimension is less then or equal to d. If the intersection is non-empty, then it can be written as

$$c \cap d = \bigtimes_{i=1}^{d} [g_{a_i}^{(i)}, g_{b_i}^{(i)}], \tag{5}$$

where $a_i = \max\{j_i, k_i\}$ and $b_i = \min\{j_i, k_i\} + 1$. Note that the i–th interval in the Cartesian product degenerates to a single point if $j_i = k_i + 1$ or $k_i = j_i + 1$ for $i = 1, \ldots, d$.

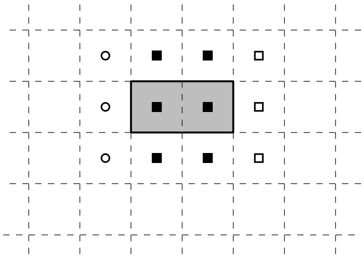
Definition 3. Let $f_c \in \Pi^{\mathbf{p}}(c)$ and $g_d \in \Pi^{\mathbf{p}}(d)$. Then we say that the polynomials f_c and g_d have a *contact* on $c \cap d$ (and write $f_c \sim g_d$) if

$$\forall \mathbf{x} \in c \cap d : \ (\partial_i^j f_c)(\mathbf{x}) = (\partial_i^j g_d)(\mathbf{x})$$

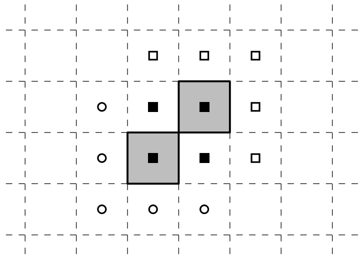
is satisfied for all $i = 1, \ldots, d$ and

$$\begin{cases} j = 0, \ldots, p^{(i)} & \text{if } a_i < b_i, \\ j = 0, \ldots, p^{(i)} - m_{a_i}^{(i)} & \text{if } a_i = b_i, \end{cases}$$

where $c \cap d$ has the form (5) and $m_{a_i}^{(i)}$ is the multiplicity of the hyperplane $G_{a_i}^{(i)}$ in the grid.



(a) edge contact



(b) vertex contact

Figure 2: Basis functions influencing two cells in biquadratic case with single knots. The circles and solid boxes correspond to basis functions influencing the left cell, while the solid boxes and empty boxes identify the basis functions which are need to represent polynomials on the second cell.

Note that any two polynomials on disjoint cells have a contact, since $c \cap d$ is empty in this case. The relation $\sim$ is symmetric and reflexive, but not transitive. The order of the contact depends on the given multiplicity of the grid hyperplanes. The higher the multiplicity, the smaller the number of derivatives which have to agree.

From now on we will use the notation

$$\beta|_{c \cap d} \neq 0$$

to express the fact the B-spline β does not vanish identically on $c \cap d$, i.e.,

$$\exists \mathbf{x} \in c \cap d : \beta(\mathbf{x}) \neq 0.$$

The contact between two polynomials on different cells can be characterized easily with the help of the B-spline coefficients.

Lemma 4 (Contact Characterisation Lemma (CCL)). *We consider two cells $c, d \in C$ with associated polynomials $f_c \in \Pi^{\mathbf{p}}(c)$ and $g_d \in \Pi^{\mathbf{p}}(d)$. These polynomials f_c and g_d have a contact on $c \cap d$ if and only if*

$$\forall \beta : \beta|_{c \cap d} \neq 0 \Rightarrow \lambda_c^\beta(f_c) = \lambda_d^\beta(g_d).$$

Proof. This can be proved by extending the univariate blossoming argument (see e.g. [24], section 7.1; the generalization for multivariate Bézier surfaces is outlined in Section 9.7) to the tensor product setting. Indeed, the two polynomials have a contact if and only if the associated values of their blossoms coincide, which then correspond to the B-spline coefficients. □

Example 5. Consider again the bivariate biquadratic case with single knots. There are three possibilities of contact between two polynomials on two cells, see Figure 2:

1. The cells are disjoint and all polynomials have a contact (not shown).

2. The cells share a vertex and the values, the two first derivatives and the mixed partial derivative are equal at this vertex (right picture).

3. The two cells share an edge (left picture) and the values, the first partial derivatives across this edge (called the cross-derivative), and all derivatives of the value and the cross-derivative along this edge take the same values. Clearly, derivatives of order higher than two along the edge are zero for biquadratic polynomials.

These situations can be characterized by the B-spline coefficients:

1. There is no condition if the cells are disjoint.
2. In the case of a vertex-vertex contact, the B-spline coefficients associated with the B-splines of the four neighboring cells have to take the same values.
3. In the case of edge-edge contact, six coefficients have to be identical.

2.3 Piecewise polynomials on multi–cell domains

We consider a finite subset $M \subset C$, which we will call a *multi-cell domain.* More precisely, the set M contains a finite number of cells of the form (2). Furthermore, we will use the abbreviation

$$\bigcup M = \bigcup_{c \in M} c$$

for the subset of $\mathbb{R}^d$ occupied by the cells from M. The set $\bigcup M$ is a closed and bounded subset of $\mathbb{R}^d$.

Definition 6. Given a multi-cell domain $M \subset C$ we define the *disconnected space* (also called *the space of piecewise polynomials*) by

$$D(M) = \{s = (s_c)_{c \in M} \mid s_c \in \Pi^{\mathbf{p}}(c)\}.$$

Thus any piecewise polynomial $s \in D(M)$ is a collection of polynomials s_c, one for each cell $c \in M$. Note that these polynomials may take different values at the grid lines, therefore it is not possible to define a global function on $\bigcup M$. Nevertheless, each of them can be represented in the B–spline basis as observed in (4),

$$s_c(\mathbf{x}) = \sum_{\beta \in B_c} \lambda_c^\beta(s_c)\beta|_c(\mathbf{x}), \quad \mathbf{x} \in c, \quad c \in M.$$

Definition 7. We consider a multi-cell domain M and the associated disconnected space $D(M)$. The *spline space* on M is defined by

$$S(M) = \{s \in D(M) \mid \forall c, d \in M : s_c \sim s_d\}. \tag{6}$$

For $s \in S(M)$ we define $\tilde{s} : \bigcup M \to \mathbb{R}$ so that

$$\tilde{s}(\mathbf{x}) = s_c(\mathbf{x}) \quad \text{if } \mathbf{x} \in c, \quad c \in M.$$

This function is well–defined (single–valued), since any two polynomial pieces s_c and s_d meet at least continuously along the intersection $c \cap d$ of the cells. By using the characteristic functions χ_c of the cells $c \in M$, we may express it in terms of the basis functions as follows:

$$\tilde{s}(\mathbf{x}) = \sum_{c \in M} \sum_{\beta \in B_c} \lambda_c^\beta(s_c)\beta(\mathbf{x})\chi_c^\star(\mathbf{x}), \quad \mathbf{x} \in \bigcup M \tag{7}$$

with the normalized characteristic functions

$$\chi_c^\star(\mathbf{x}) = \begin{cases} \dfrac{\chi(\mathbf{x})}{\sum_{c\in M}\chi_c(\mathbf{x})}, & \text{if} \quad \mathbf{x} \in \bigcup M. \\ 0 \quad \text{otherwise} \end{cases}$$

Strictly speaking, the elements of $S(M)$ are $|M|$–tuples of polynomials, where $|M|$ is the number of cells in M. In order to keep the notation simple, we will use the same notation for the actual spline functions $\tilde{s}$.

That is, we will consider the elements of $S(M)$ simultaneously as $|M|$–tuples of polynomials and as piecewise polynomial functions defined on $\bigcup M$. Consequently, we will simply write s instead of $\tilde{s}$, and we will denote the *linear space of all piecewise polynomial functions on* $\bigcup M$ *with the required contacts* between the polynomial segments as $S(M)$.

Definition 8. For each basis function $\beta \in B$ we now define the *coefficient graph* $\mathcal{G}_\beta$ as follows.

- The vertices of $\mathcal{G}_\beta$ are the cells $c \in M$ such that $c \subset \overline{\text{supp}\,\beta}$.
- Two vertices c and d are connected by an edge if $\beta|_{c\cap d} \neq 0$.

The set of connected components of this graph will be denoted with $CC(\mathcal{G}_\beta)$.

If there is no overlap of β with $\bigcup M$ then both the coefficient graph $\mathcal{G}_\beta$ and the set $CC(\mathcal{G}_\beta)$ of connected components are empty.

Example 9. We consider again the bivariate biquadratic case. Figure 3 shows a multi-cell domain consisting of eight cells (a) and the supports of three basis functions (b). The coefficient graphs of these three basis functions are presented in (c). The coefficient graphs of β and γ have only one connected component, while the coefficient graph of α has got two of them.

Proposition 10. *Consider a piecewise polynomial $s \in D(M)$. Then s is contained in the spline space $S(M)$ if and only if $\lambda_c^\beta(s_c) = \lambda_d^\beta(s_d)$ whenever c and d belong to the same connected component of $\mathcal{G}_\beta$.*

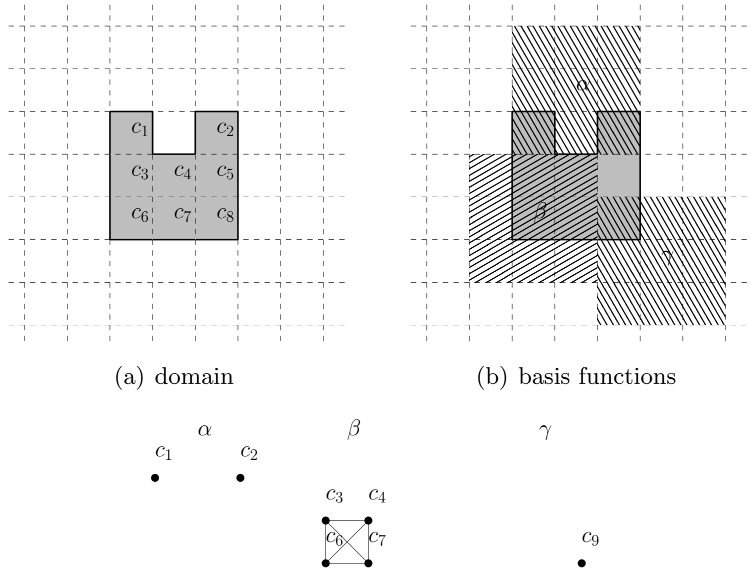
Proof. Consider a piecewise polynomial $s \in S(M)$. Assume there exist c, d in the same connected component such that $\lambda_c^\beta(s_c) \neq \lambda_d^\beta(s_d)$. Then there exist two different values of coefficients corresponding to neighboring vertices c' and d' in the connected component. According to CCL (Lemma 4) $s_{c'}$ and $s_{d'}$ do not have contact and therefore s does not belong to $S(M)$.

On the other hand, if all coefficients $\lambda_c^\beta(s_c)$ for all cells c belonging to one connected component of $\mathcal{G}_\beta$ take the same value, then all s_c have a contact by CCL (Lemma 4) and thus $s \in S(M)$. □

2.4 Basis of splines on multi–cell domains

Definition 11. For every $\beta \in B$ and every $H \in CC(\mathcal{G}_\beta)$ we define the function

$$\beta_H(\mathbf{x}) = \sum_{c\in H} \beta(\mathbf{x})\chi_c^\star(\mathbf{x}).$$



(a) domain (b) basis functions

(c) coefficient graphs

Figure 3: A multi-cell domain with eight cells (a), the supports of three biquadratic B-splines (b) and the associated coefficient graphs (c).

The set of all these functions is denoted by

$$\Delta_B = \bigcup_{\beta \in B} \{\beta_H \mid H \in CC(\mathcal{G}_\beta)\}.$$

Theorem 12. *The set Δ_B – when restricted to $\bigcup M$ – forms a locally linearly independent basis of $S(M)$.*

Proof. Consider $s \in S(M)$. First we prove that s can be obtained as a linear combination of functions from Δ_B. Rearranging (7) gives

$$s(\mathbf{x}) = \sum_{\beta \in B} \sum_{c \in \mathcal{G}_\beta} \lambda_c^\beta(s_c)\beta(\mathbf{x})\chi_c^\star(\mathbf{x}) = \sum_{\beta \in B} \sum_{H \in CC(\mathcal{G}_\beta)} \sum_{c \in H} \lambda_c^\beta(s_c)\beta(\mathbf{x})\chi_c^\star(\mathbf{x}), \qquad (8)$$

where $\mathbf{x} \in \mathbb{R}^d$. For each function $\beta \in B$ and for each $H \in CC(\mathcal{G}_\beta)$ all the coefficients $\lambda_c^\beta(s_c)$ have to be the same for all $c \in H$ according to Proposition 10. We will denote this coefficient by $\lambda_H^\beta(s)$. Thus we may rewrite (8) as

$$s(\mathbf{x}) = \sum_{\beta \in B} \sum_{H \in CC(\mathcal{G}_\beta)} \lambda_H^\beta(s) \underbrace{\sum_{c \in H} \beta(\mathbf{x})\chi_c^\star(\mathbf{x})}_{=\beta_H}.$$

Second we prove the local linear independence of the functions. Consider an open subset $X \subset \bigcup M$ and a linear combination of functions β_H that do not vanish on X, which is equal to zero on X. For each β_H we consider a cell $c \in H$, which has a nonempty intersection

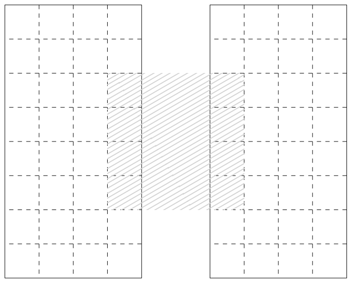


Figure 4: The support of a bicubic B-spline basis which violates the assumption of Corollary 13 with respect to the domain consisting of all shown cells.

with X. Clearly, β_H does not vanish on c. Moreover, the restrictions of all functions β_H to this cell are either zero or equal to the restrictions of mutually different tensor–product B–splines $\beta \in B$. From the local linear independence of functions $\beta \in B$ we then obtain that the coefficient of β_H in the linear combination is zero. Repeating this for all functions β_H we conclude that the functions β_H are locally linearly independent. Clearly, this also implies the linear independence of Δ_B. □

Corollary 13. *If each of the intersections of the supports* supp β *with the multi–cell domain* $\bigcup M$ *is connected, then the functions in*

$$B_M = \{\beta \in B \mid \operatorname{supp} \beta \cap \left(\bigcup M\right) \neq \emptyset\},$$

when restricted to $\bigcup M$*, form a basis of* $S(M)$*.*

Proof. Indeed, if this condition is satisfied, then each coefficient graph in Theorem 12 has either one connected component or it is empty. □

Example 14. The condition concerning the connected sets in Corollary 13 means that there is no situation as shown in Figure 4 for bicubic B-splines with single knots.

3 Hierarchical splines

We use the notations from the previous section in a hierarchical setting and define a hierarchical spline space and a hierarchical basis. Finally we prove that the hierarchical basis indeed spans the entire hierarchical spline space.

3.1 Hierarchies of tensor-product spline spaces

In order to define a hierarchical tensor-product spline space, we need to introduce a hierarchy of tensor-product spline spaces and a hierarchy of domains.

First we consider the spline spaces. Given a maximum level N, we consider a sequence of grids G^ℓ, $\ell = 0, \ldots, N$ with associated degrees $\mathbf{p}^\ell = (p_\ell^{(1)}, \ldots, p_\ell^{(d)})$ where we assume that the *degrees do not decrease*,

$$\mathbf{p}^\ell \leq \mathbf{p}^{\ell+1}, \quad \text{i.e.} \quad p_\ell^{(i)} \leq p_{\ell+1}^{(i)}, \quad i = 0, \ldots, d;\ \ell = 0, \ldots, N-1 \tag{9}$$

Each grid hyperplane $G_{j,\ell}^{(i)} \in G^\ell$ has an associated multiplicity $m_{j,\ell}^{(i)}$ which satisfies the assumption (1) level by level.

We assume that the *grids are nested* in the following sense. Every grid hyperplane in G^ℓ is also present in $G^{\ell+1}$ and its multiplicity in the higher level is at least equal to the previous multiplicity plus the increase of the degree in the corresponding coordinate direction.

Based on the sequence of grids and degrees, we now define on each grid G^ℓ the set of tensor–product B–splines B^ℓ of degree $\mathbf{p}^\ell$. The span of the B-splines defines spline spaces span $B^\ell, \ell = 0, \dots N$. Under the previous assumptions concerning non-decreasing degrees and nested grids, the linear spaces spanned by the B-splines are nested,

$$\text{span } B^\ell \subseteq \text{span } B^{\ell+1}, \quad \ell = 0, \dots, N-1.$$

For each level ℓ, the grid G^ℓ and the degrees $\mathbf{p}^\ell$ allow to apply the theory from the previous section. Thus, given a multi-cell domain M^ℓ with respect to the grid G^ℓ, one may define a spline space $S^\ell(M^\ell)$. The connected components of the B-splines B^ℓ with respect to $\bigcup M^\ell$ form a basis of this space, according to Theorem 12.

Second we consider a *nested sequence of domains*

$$\Omega^0 \supseteq \Omega^1 \supseteq \dots \supseteq \Omega^N = \emptyset.$$

These domains are required to satisfy the following condition.

Assumption 15. *We assume that each set*

$$\Omega^0 \setminus \Omega^{\ell+1}, \quad \ell = 0 \dots N-1,$$

can be represented as a multi–cell domain with respect to grid G^ℓ. More precisely, we assume that there exists a multi-cell domain $M^\ell \subseteq C^\ell$, which is a finite set of cells of the grid G^ℓ, satisfying

$$\Omega^0 \setminus \Omega^{\ell+1} = \bigcup M^\ell.$$

From now on we will use M^ℓ to denote this multi-cell domain. For convenience, we set

$$\bigcup M^{-1} = \emptyset.$$

The sets $\bigcup M^\ell = \Omega^0 \setminus \Omega^{\ell+1}$ were denoted as *rings* R^ℓ in [12], because, conceptually, they represent the domain Ω^0 with the "hole" $\Omega^{\ell+1}$. We will also adopt this notion. The above assumption concerning the shape of the rings is actually weaker than the one in [12], where each Ω^ℓ was assumed to be a multi–cell domain of level $\max(0, \ell-1)$.

It should be noted that these rings are also nested,

$$\Omega^0 = \bigcup M^{N-1} \supseteq \bigcup M^{N-2} \supseteq \dots \supseteq \bigcup M^0 \supseteq \bigcup M^{-1} = \emptyset. \qquad (10)$$

Based on the sequences of function spaces and domains we are now able define the hierarchical spline space by the property, that the restriction of a function to each of the multi-cell domains $\bigcup M^\ell$ belongs to the corresponding spline space $S^\ell(M^\ell)$:

Definition 16. The *hierarchical spline space* H is given by

$$H = \{h : \Omega^0 \to \mathbb{R} \mid \forall \ell : h|_{\bigcup M^\ell} \in S^\ell(M^\ell)\}.$$

The next section discusses the existence of a B-spline basis for this space.

3.2 The basis of the hierarchical spline space

Recall that we defined a tensor–product spline basis B^ℓ on each grid G^ℓ. Similarly to (3) we consider the B-splines whose support intersects the ring $\bigcup M^\ell$,

$$B^\ell_{M^\ell} := \{\beta \in B^\ell \mid \text{supp}\,\beta \cap \left(\bigcup M^\ell\right) \neq \emptyset\}.$$

Based on the definition of the rings, we again use the selection procedure from [12] which slightly generalizes the earlier method proposed by Kraft in [17] by also allowing for coinciding subdomain boundaries.

Definition 17. The *hierarchical basis* $\mathcal{K}$ is defined as

$$\mathcal{K} = \bigcup_{\ell=0}^{N-1} \mathcal{K}^\ell,$$

with

$$\mathcal{K}^\ell = \{\beta \in B^\ell_{M^\ell} \mid \text{supp}\,\beta \cap \left(\bigcup M^{\ell-1}\right) = \emptyset\}.$$

It can be show that this set of B-splines is indeed linearly independent, see [17] or [27]. This follows directly from the local linear independence of the individual bases $\mathcal{K}^\ell \subset B^\ell$.

Now we are able to formulate the main result of this paper.

Theorem 18. *If the assumption of Corollary 13 is satisfied for each level ℓ, i.e., provided that all sets*

$$\text{supp}\,\beta \cap \left(\bigcup M^\ell\right), \quad \ell = 0, \ldots, N-1; \quad \beta \in B^\ell,$$

are connected, then the hierarchical spline basis $\mathcal{K}$ from Definition 17 spans the entire space H.

Proof. The proof is very similar to the proof of Theorem 20 in [12]. Nevertheless, in order to make this paper self–contained, we repeat it here with the new notation.

We show that any function $h \in H$ can be obtained as a linear combination of functions from $\mathcal{K}$, i.e., $h \in \text{span}\,\mathcal{K}|_{\Omega^0}$. This is proved in three steps.

- Step 1: There exist N functions

$$h^\ell \in \text{span}\,B^\ell_{M^\ell} \quad (\ell = 0, \ldots, N-1) \tag{11}$$

such that

$$h^\ell|_{\bigcup M^\ell} = \left(h - \sum_{i=0}^{\ell-1} h^i\right)|_{\bigcup M^\ell}. \tag{12}$$

This we prove by induction:

For $\ell = 0$ condition (12) reads $h^0|_{\bigcup M^0} = h|_{\bigcup M^0}$. But since $h|_{\bigcup M^0} \in S^0(\bigcup M^0)$ and the required assumption for $\bigcup M^0$ is fulfilled, validity of (12) is granted by Corollary 13.

Now we consider a general value of ℓ. This assumption also holds for $\bigcup M^\ell$, so $h|_{\bigcup M^\ell} \in \text{span}\,B^\ell|_{\bigcup M^\ell}$ by Corollary 13. In addition, due to the induction assumption we have

$$h^i \in \text{span}\,B^i_{M^i} \subseteq \text{span}\,B^i,$$

hence also

$$h^i|_{\bigcup M^\ell} \in \text{span}\,B^i|_{\bigcup M^\ell} \subseteq \text{span}\,B^\ell|_{\bigcup M^\ell} = \text{span}\,B^\ell_{M^\ell}|_{\bigcup M^\ell} \subseteq S^\ell(M^\ell), \quad i = 0, \ldots, \ell-1.$$

This implies (11).

- Step 2: These functions satisfy

$$h^\ell|_{\bigcup M^{\ell-1}} = 0, \quad \ell = 0, \ldots, N. \tag{13}$$

Rewriting (12) gives

$$h|_{\bigcup M^\ell} = \sum_{i=0}^{\ell} h^i|_{\bigcup M^\ell} \text{ and also } h|_{\bigcup M^{\ell-1}} = \sum_{i=0}^{\ell} h^i|_{\bigcup M^{\ell-1}} \tag{14}$$

since $\bigcup M^{\ell-1} \subseteq \bigcup M^\ell$. Rewriting (12) with $h^{\ell-1}$ gives

$$h|_{\bigcup M^{\ell-1}} = \sum_{i=0}^{\ell-1} h^i|_{\bigcup M^{\ell-1}},$$

which, when compared with (14), gives the desired result.

- Step 3: These functions satisfy

$$h^\ell \in \operatorname{span} \mathcal{K}^\ell = \operatorname{span} \{\beta \in B^\ell_{M^\ell} \mid \operatorname{supp} \beta \cap \left(\bigcup M^{\ell-1}\right) = \emptyset\} \tag{15}$$

To see this, consider representation of h granted by (11):

$$h^\ell = \sum_{\beta \in B^\ell_{M^\ell}} c_\beta \beta.$$

Equation (13) from Step 2 together with the local linear independence of basis functions in $B^\ell_{M^\ell}$ implies that for any function β with a nonempty intersection with $\bigcup M^{\ell-1}$, the coefficient c_β assigned to it is necessarily zero. This proves (15).

Finally, rewriting (12) for $\ell = N-1$ into the form

$$h|_{\bigcup M^{N-1}} = \sum_{i=0}^{N-1} h^i|_{\bigcup M^{N-1}}$$

concludes the proof, since $\bigcup M^{N-1} = \Omega^0$, $h^i \in \operatorname{span} \mathcal{K}^i$ and $\bigcup_{i=0}^{N-1} \mathcal{K}^i = \mathcal{K}$. □

The condition of Theorem 18 is satisfied if each subdomain Ω^ℓ is either sufficiently small or sufficiently large with respect to the supports of the B-splines at the previous level. This condition is slightly weaker than the condition which was used in [12].

Example 19. Consider the case where all degrees (at all levels and in all coordinate directions) are equal to p and all multiplicities of hyperplanes are equal to 1. In this case, the condition is satisfied, if each $\Omega^0 \setminus \Omega^\ell$ admits a self-intersection-free offset at distance $p/2$ with respect to the grid $G^{\ell-1}$, or if each Ω^ℓ is contained in a box consisting of $(p-1) \times \cdots \times (p-1)$ cells of the grid $G^{\ell-1}$, or if each Ω^ℓ is a disjoint union of sets possessing either property.

4 Closure

We analyzed the dimensions and the bases of multivariate tensor-product spline functions on a multi-cell domain. Based on theses results, by a slight generalization of the techniques from [12], we derived a simple sufficient condition for the completeness of a hierarchical spline space. More precisely, this condition guarantees that any piecewise polynomial functions on the given hierarchical grid can be represented in the hierarchical tensor-product B-spline basis.

Further work will focus on formulas for the dimensions for the spline spaces, which are currently only given implicitly by the number of connected B-spline components, and on the use of the truncated hierarchical B-spline basis, which should lead to weaker sufficient conditions for the completeness of the hierarchical spline spaces.

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