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A. Bressan, B. Jüttler

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A Hierarchical Construction of LR Meshes in 2D

Andrea Bressan *and* Bert Jüttler

Institute of Applied Geometry,
Johannes Kepler University of Linz,
Altenberger Str. 69, 4040 Linz, Austria

Abstract

We describe a construction of LR-spaces whose bases are composed of locally linearly independent B-splines which also form a partition of unity. The construction conforms to given refinement requirements associated to subdomains. In contrast to the original LR-paper (Dokken et al., 2013) and similarly to the hierarchical B-spline framework (Forsey and Bartels, 1988) the construction of the mesh is based on a priori choice of a sequence of nested tensor B-spline spaces.

1 Introduction

In the last decade the use of spline spaces has spread from the field of applied geometry, in particular Computer Aided Design (CAD), to that of numerical analysis of Partial Differential Equations (PDE). This is largely due to the influence of the seminal paper by Hughes et al. (2005). The use of B-spline generated spaces in Galerkin methods was attempted before by Höllig (2003), but Hughes et al. (2005) recognized it as a possible way to remove the compatibility layer that is in between the CAD tools and the Finite Element Method (FEM). The compatibility layer contains the mesh generation process and in some cases can be more computationally expensive than the simulation itself (Hughes et al., 2005). The method that reduces the compatibility layer proposed by Hughes et al. (2005) is called IsoGeometric Analysis

(IGA) and is based on the isoparametric approach: the solution fields of the PDE are in the same B-spline or NURBS space used for the parametrization of the geometry.

IGA sprouted new research in numerical methods due to the availability of basis functions with higher smoothness and with strong algebraic properties that allow for new numerical schemes like compatible discretizations. It had the same effect in the applied geometry field: the numerical simulation of PDEs requires high quality parametrizations of the domain while in CAD it is common to parametrize only the boundary and to allow both for small gaps and singularities.

Both CAD and IGA applications require the use of function spaces that allow for local changes in spatial resolution. This is necessary to obtain a good approximation with fewer degrees of freedom. The standard tensor-product B-spline spaces do not allow for local changes in spatial resolution and thus different generalizations providing adaptive refinement were proposed in the last 25 years. Forsey and Bartels (1988) introduced the hierarchical-splines, later studied by Kraft (1997) and more recently by Giannelli et al. (2012) and Mokriš et al. (2014); Sederberg et al. (2003, 2004) introduced T-splines of which an Analysis Suitable subset (AST) was described by Beirão da Veiga et al. (2012); Deng et al. (2008) introduced PHT-splines and Dokken et al. (2013) introduced LR-splines whose local linear independence was studied by Bressan (2013). Each of these approaches has their own strengths and weaknesses determined by the focus with which they were developed. In this article we try to combine the LR-splines framework with the hierarchical approach.

Our aim is to obtain a space that has strong properties such as local linear independence and that can be efficiently implemented. Johannessen et al. (2014) applied LR-spline spaces to IGA and explored different refinement techniques. In contrast to their work, we study refinement strategies that are based on theoretical guarantees. In detail we present a method to construct a box mesh $\mathcal{M}$ on a domain Ω whose element size is small in a neighborhood of some given regions and for which the associated LR-spline collection $\mathcal{LR}(\mathcal{M})$ is a basis composed of locally linearly independent functions. This implies that the basis is also a partition of unity.

In Section 2 we recall LR-spline definitions and results. In particular we focus on the equivalence for the LR-spline collection to be a partition of unity, to be a set of locally linearly independent B-splines, and the *non-nested support property* (N_2S for short).

In Section 3 we describe a subset of the domain Ω in which it is possible to add vertical segments while preserving the N_2S property. We describe another subset that behaves similarly for the addition of horizontal segments.

In Section 4 we define a hierarchical approach to the construction of box meshes. Then we provide sufficient conditions under which the associated LR-spline space has the N_2S property.

In Section 5 we study the completeness of the hierarchically constructed LR-space, that is, whether it equals the piecewise polynomial space that is associated to the mesh.

Section 6 describes our construction of LR meshes that guarantees both the N_2S property (and thus local linear independence of the basis functions) and completeness. We comment on the locality of the refinement and show some examples in the case of dyadic refinement.

Section 7 compares the proposed space with the truncated hierarchical B-spline space (THB) on the same Bézier mesh.

2 Notation and LR-spline properties

We use $\mathbb{P}_d$ to denote the space of polynomials of degree less than or equal to d. The space of bivariate polynomials of degree d_x in the x variable and degree d_y in the y variable is denoted using a vector $\mathbf{d} = (d_x, d_y)$ for the degree:

$$\mathbb{P}_{\mathbf{d}} = \mathbb{P}_{d_x} \otimes \mathbb{P}_{d_y}.$$

For our purpose the degree $\mathbf{d} = (d_x, d_y)$ can be considered fixed at the beginning and it will be omitted in the notation.

A *knot vector* Θ is a monotone non-decreasing sequence of real numbers

$$\theta_1 \leq \ldots \leq \theta_n.$$

The number of repetitions of a knot z in a knot vector Θ is called the *multiplicity* of z in Θ and denoted with:

$$\mu_\Theta(z) = \#\{j : \theta_j = z\}. \tag{1}$$

We say that two knot vectors $\theta_1 \leq \ldots \leq \theta_n$ and $\xi_1 \leq \ldots \leq \xi_m$ are *compatible on the overlap*[1] if they can be seen as two parts of a larger knot vector

[1]In the context of T-splines this property is called *overlap* (Beirão da Veiga et al., 2012). We prefer the name *compatible* because two separate knot vectors with $\theta_n < \xi_1$ or $\xi_m < \theta_1$ are *compatible*, but it would be counter-intuitive to call them *overlapping*.

$\zeta_1 \leq \ldots \leq \zeta_{m+n}$. More precisely if there exists $\zeta_1 \leq \ldots \leq \zeta_{m+n}$, and two indexes s, t such that

$$\begin{cases} \theta_i = \zeta_{s+i}, & i = 1, \ldots, n; \\ \xi_i = \zeta_{t+i}, & i = 1, \ldots, m. \end{cases} \tag{2}$$

Note that if $\theta_n \leq \xi_1$ or if $\xi_m \leq \theta_1$ than the two not vectors are compatible.

The B-spline of degree d defined by the knot vector $\Theta = (\theta_1, \ldots, \theta_{d+2})$ (with $\theta_1 < \theta_{d+2}$) is denoted with $B[\Theta]$. Bi-variate B-splines $\mathbb{R}^2 \to \mathbb{R}$ are the product of two univariate B-splines and are defined by a pair of knot vectors $\boldsymbol{\Theta} = (\Theta_x, \Theta_y)$

$$B[\boldsymbol{\Theta}](x, y) = B[\Theta_x](x)B[\Theta_y](y).$$

The definition of the LR-spline spaces is based on knot insertion. Given a knot vector $\Theta = (\theta_1, \ldots, \theta_{k+2})$ and $\bar{\theta} \in]\theta_1, \theta_{k+2}[$ it is possible to insert $\bar{\theta}$ in Θ and obtain the two following knot vectors

$$\Theta^+ = (\theta_1, \ldots, \bar{\theta}, \ldots, \theta_{k+1}),$$
$$\Theta^- = (\theta_2, \ldots, \bar{\theta}, \ldots, \theta_{k+2}).$$

There is a linear relation involving the B-splines $B[\Theta]$, $B[\Theta^+]$ and $B[\Theta^-]$. In the bivariate setting, given $B[\Theta_x, \Theta_y]$ and $\bar{\theta}$ in $[\theta_{x,i}, \theta_{x,i+1}[$ it holds

$$B[\boldsymbol{\Theta}] = \alpha^+ B[\Theta_x^+, \Theta_y] + \alpha^- B[\Theta_x^-, \Theta_y]. \tag{3}$$

where

$$\alpha^+ = \begin{cases} 1 & i = d_x, \\ \frac{\bar{\theta} - \theta_{x,1}}{\theta_{x,d_x+1} - \theta_{x,1}} & \text{otherwise,} \end{cases} \qquad \alpha^- = \begin{cases} 1 & i = 1, \\ \frac{\theta_{x,d_x+2} - \bar{\theta}}{\theta_{x,d_x+2} - \theta_{x,2}} & \text{otherwise.} \end{cases}$$

We say that $B[\Theta_x^+, \Theta_y]$ and $B[\Theta_x^-, \Theta_y]$ are obtained from $B[\Theta_x, \Theta_y]$ by the insertion of $\bar{\theta}$. The insertion of knots in Θ_y works similarly. Based on the knot insertion is the notion of *nested* B-splines.

Definition 1. Let S be a set of functions. A B-spline B is *nested* in a B-spline B' relatively to S and it is written $B \prec_S B'$ if there exists a sequence of B-splines $B' = B_1, \ldots, B_n = B$ such that:

- $B_i \in S$, for $i = 1, \ldots, n$;
- B_{i+1} is obtained from B_i by the insertion of a knot, $i = 1, \ldots, n-1$.

If S is the set of all B-spline functions $\mathbb{R}^2 \to \mathbb{R}$, then it will be omitted and we will use $\prec$ instead of $\prec_S$.

Note that $\prec_S$ is a partial order relation on B-splines. As such it is possible to describe $\prec_S$ using a directed acyclic graph. Minimal and maximal elements of S with respect of $\prec_S$ correspond to sinks and sources of the graph. *Comparable pairs*, i.e. pairs B_1, B_2 such that $B_1 \prec B_2$ or $B_2 \prec B_1$, corresponds to pairs of elements that are connected by an oriented path in the graph.

It is important to note that $B \prec_S B'$ implies $B \prec B'$, but not vice versa. In particular if $B \in S$ is a maximal or a minimal element with respect to $\prec$ then it is also a maximal or minimal element for $\prec_S$. The minimal elements for $\prec_S$ in S are called *minimal support* B-splines in S.

The definition of LR-spline spaces is based on $\prec_S$ where S is an appropriate piecewise polynomial space over box elements. A box η is a Cartesian product of two closed intervals: $[a,b] \times [e,f]$. A box in $\mathbb{R}^2$ can be:

- a *vertex* if $a = b$ and $e = f$;
- a *horizontal segment* if $a < b$ and $e = f$;
- a *vertical segment* if $a = b$ and $e < f$;
- a *rectangle* if $a < b$ and $e < f$.

By convention the interior of a horizontal or vertical segment $\gamma = [a,b] \times [e,f]$ is

$$\gamma^\circ = \gamma \setminus \{(a,e),(b,f)\}. \tag{4}$$

The interior of a rectangle $\gamma = [a,b] \times [e,f]$ is its topological interior $\gamma^\circ =]a,b[\times]e,f[$ and the interior of a vertex is the empty set.

A *box mesh* $\mathcal{R}$ on a rectangle Ω is a finite collection of rectangles such that:

- $\bigcup_{\eta \in \mathcal{R}} \eta = \Omega$;
- $\forall \eta_1 \neq \eta_2 \in \mathcal{R}$, ${\eta_1}^\circ \cap {\eta_2}^\circ = \emptyset$.

For each box mesh we define the sets of vertical and horizontal *edges*,

$$\mathcal{E}^v(\mathcal{R}) = \{\gamma = \eta_1 \cap \eta_2 : \eta_1, \eta_2 \in \mathcal{R} \text{ and } \gamma \text{ is a vertical segment}\}$$
$$\mathcal{E}^h(\mathcal{R}) = \{\gamma = \eta_1 \cap \eta_2 : \eta_1, \eta_2 \in \mathcal{R} \text{ and } \gamma \text{ is a horizontal segment}\}.$$

Their union is the set of all edges

$$\mathcal{E}(\mathcal{R}) = \mathcal{E}^v(\mathcal{R}) \cup \mathcal{E}^h(\mathcal{R}).$$

Definition 2. A *box mesh with multiplicity* $\mathcal{M}$ on Ω is a pair $(\mathcal{R}, \sigma)$ where $\mathcal{R}$ is box mesh and $\sigma : \mathcal{E}(\mathcal{R}) \to \mathbb{N}$ is a function representing the multiplicity of the edges. The spline space of degree $\mathbf{d}$ associated to a box mesh with multiplicity is

$$\mathbb{S}(\mathcal{M}) = \left\{ \begin{array}{l} f : \Omega \to \mathbb{R} : \forall \eta \in \mathcal{R},\ f|_\eta \in \mathbb{P}_{\mathbf{d}}, \\ \qquad\qquad \forall \gamma \in \mathcal{E}^v(\mathcal{M}),\ \partial_x^{d_x - \sigma(\gamma)} f \text{ is continuous on } \gamma^\circ, \\ \qquad\qquad \forall \gamma \in \mathcal{E}^h(\mathcal{M}),\ \partial_y^{d_y - \sigma(\gamma)} f \text{ is continuous on } \gamma^\circ \end{array} \right\}.$$

We say that a B-spline $B : \mathbb{R}^2 \to \mathbb{R}$ is in $\mathbb{S}(\mathcal{M})$ when $\operatorname{supp} B \subseteq \Omega$ and $B|_\Omega \in \mathbb{S}(\mathcal{M})$. When the component of $\mathcal{M}$ are not specified we will refer to $\mathcal{E}(\mathcal{R})$ by $\mathcal{E}(\mathcal{M})$.

Throughout this paper we will assume that the multiplicity of horizontal and vertical edges does not exceed $d_y + 1$ and $d_x + 1$, respectively.

Box meshes with multiplicity can be refined by adding mesh-lines to them. Given a mesh $\mathcal{M} = (\mathcal{R}, \sigma)$ and a segment γ we define $\mathcal{M} + \gamma$ as the mesh $\tilde{\mathcal{M}} = (\tilde{\mathcal{R}}, \tilde{\sigma})$ obtained by dividing all rectangles split by γ into two. More precisely

$$\tilde{\mathcal{R}} = \left\{ \overline{C} : \exists \eta \in \mathcal{R} :\ C \text{ is a connected component of } \eta \setminus \gamma \right\} \tag{5}$$

and for $\alpha \in \mathcal{E}(\tilde{\mathcal{R}})$

$$\tilde{\sigma}(\alpha) = \begin{cases} \sigma(\beta) & \alpha \not\subseteq \gamma \wedge \exists \beta \in \mathcal{E}(\mathcal{M}) : \alpha \subseteq \beta \\ \sigma(\beta) + 1 & \alpha \subseteq \gamma \wedge \exists \beta \in \mathcal{E}(\mathcal{M}) : \alpha \subseteq \beta \\ 1 & \alpha \subseteq \gamma \wedge \nexists \beta \in \mathcal{E}(\mathcal{M}) : \alpha \subseteq \beta. \end{cases} \tag{6}$$

For all meshes $\mathcal{M}$ the space $\mathbb{S}(\mathcal{M})$ contains the Bernstein polynomials on Ω. Moreover they are always maximal elements in $\mathbb{S}(\mathcal{M})$ for both $\prec$ and $\prec_{\mathbb{S}(\mathcal{M})}$.

Definition 3. The *LR-spline collection* $\mathcal{LR}(\mathcal{M})$ is the set of the minimal support B-spline that are comparable with respect to $\prec_{\mathbb{S}(\mathcal{M})}$ to at least one Bernstein polynomial. The *LR-spline space* is

$$\mathbb{LR}(\mathcal{M}) = \operatorname{span} \mathcal{LR}(\mathcal{M}).$$

With this definition the LR-spline collection can be constructed using a recursive algorithm that, starting from the set of Bernstein polynomials, replaces at each step a non minimal support B-spline with two B-splines obtained by knot insertion. This iterative construction is described in detail by Dokken et al. (2013). The set of minimal support B-splines can indeed be larger than $\mathcal{LR}$, see for an example Figure 2 of (Bressan, 2013). The question of *completeness* of the LR-space, i.e. if $\mathbb{LR}(\mathcal{M})$ is $\mathbb{S}(\mathcal{M})$ is not trivial. Dokken et al. (2013) described sufficient conditions for the equality, these are the base of our discussion on completeness.

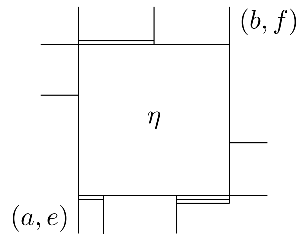
Linear independence of the B-splines in $\mathcal{LR}(\mathcal{M})$ is also a not fully resolved issue. Dokken et al. (2013) provided an algorithm that allows to check for linear relation efficiently, but in the literature there is no (non-trivial) construction that guarantees linear independence. The construction we describe is based on the theoretical results from Bressan (2013) where it is proved that local linear independence is equivalent to the fact that all B-splines in $\mathcal{LR}(\mathcal{M})$ are minimal elements with respect to $\prec$, see Theorem 4 below for details.

Theorem 4. *Let $\mathcal{M}$ be an LR mesh, then the following are equivalent:*

1. $\forall B_1, B_2 \in \mathcal{LR}(\mathcal{M}) : B_1 \prec B_2 \Rightarrow B_1 = B_2$;
2. $\forall f \in \mathbb{P}_{\mathbf{d}}$, $f = \sum_{B\in\mathcal{LR}} \mathfrak{B}[f](B)B$ *where* $\mathfrak{B}[f](B)$ *is the blossom of* f *evaluated at the internal knots of* B *(Ramshaw, 1989);*
3. $\mathcal{LR}(\mathcal{M})$ *is a partition of unity;*
4. *the functions in* $\mathcal{LR}(\mathcal{M})$ *are locally linearly independent;*
5. $\forall \eta \in \mathcal{R}$, $\#\{B \in \mathcal{LR}(\mathcal{M}) : \operatorname{supp} B \supseteq \eta^\circ\} = (d_x + 1)(d_y + 1)$.

The property described in point 1 of this theorem will be called the *non-nested support (N_2S) property*. The support can actually be nested in the physical space (this requires that some mesh edges have multiplicity > 1), but they can not be nested in the "index space". Box meshes with multiplicity for which $\mathcal{LR}$ has the N_2S property will be called *N_2S meshes*.

In the following sections we will avoid specifying the mesh when there is no ambiguity: we will use the shorter notation $\mathbb{S}$ instead of $\mathbb{S}(\mathcal{M})$, similarly $\mathcal{E}$ for $\mathcal{E}(\mathcal{M})$ etc.



$s_\eta = 3,$
$t_\eta = 2,$
$\Xi_\eta = (e, e, e, f, f).$

Figure 1: An element η with the associated s_η, t_η and Ξ_η.

3 Addition of segments

In this section we describe the set $\mathcal{R}_x$ of *horizontaly refinable rectangles*. Our result is that if $\mathcal{M}$ is a N_2S mesh and γ is a vertical segment "well contained" in $\bigcup \mathcal{R}_x$ then $\mathcal{M} + \gamma$ is a N_2S mesh[2]. Similarly we define the set $\mathcal{R}_y$ of *vertically refinable rectangles*. Here "well contained" means that only the vertices of γ can be in $\partial \bigcup \mathcal{R}_x$ or equivalently that $\gamma^\circ \subset (\bigcup \mathcal{R}_x)^\circ$. We also provide additional conditions that guarantee that the N_2S property is preserved for the limit case: when the intersection of γ with the boundary of $\bigcup \mathcal{R}_x$ is a union of segments (and similarly for horizontal γ).

Definition 5. Consider a rectangle $\eta = [a, b] \times [e, f] \in \mathcal{R}$ and let

$$s_\eta = \max\{\sigma(\gamma) : \gamma \in \mathcal{E}^h \wedge \gamma \subseteq [a, b] \times \{e\}\}$$
$$t_\eta = \max\{\sigma(\gamma) : \gamma \in \mathcal{E}^h \wedge \gamma \subseteq [a, b] \times \{f\}\}$$
$$\Xi_\eta = \big(\underbrace{e, \dots, e}_{s_\eta \text{ times}}, \underbrace{f, \dots, f}_{t_\eta \text{ times}}\big).$$

See Figure 1 for a graphical representation. We say that $\eta \in \mathcal{R}$ is *horizontally refinable* if for all $B[\Theta_x, \Theta_y] \in \mathcal{LR}(\mathcal{M})$ with $\operatorname{supp} B[\Theta_x, \Theta_y] \supseteq \eta$ it holds that Θ_y is compatible with Ξ_η. The set of horizontally refinable rectangles is $\mathcal{R}_x$. The set $\mathcal{R}_y$ of *vertically refinable rectangles* is defined similarly.

Note that from the definition it follows that for all $\eta = [a, b] \times [e, f] \in \mathcal{R}_x$ and $\alpha \in \mathcal{E}^h$ contained in $[a, b] \times \{e\}$ it holds that $\sigma(\alpha) = s_\eta$. Similarly if $\alpha \subseteq [a, b] \times \{f\}$ then $\sigma(\alpha) = t_\eta$

[2]We use the convention that $\bigcup A = \bigcup_{a \in A} a$.

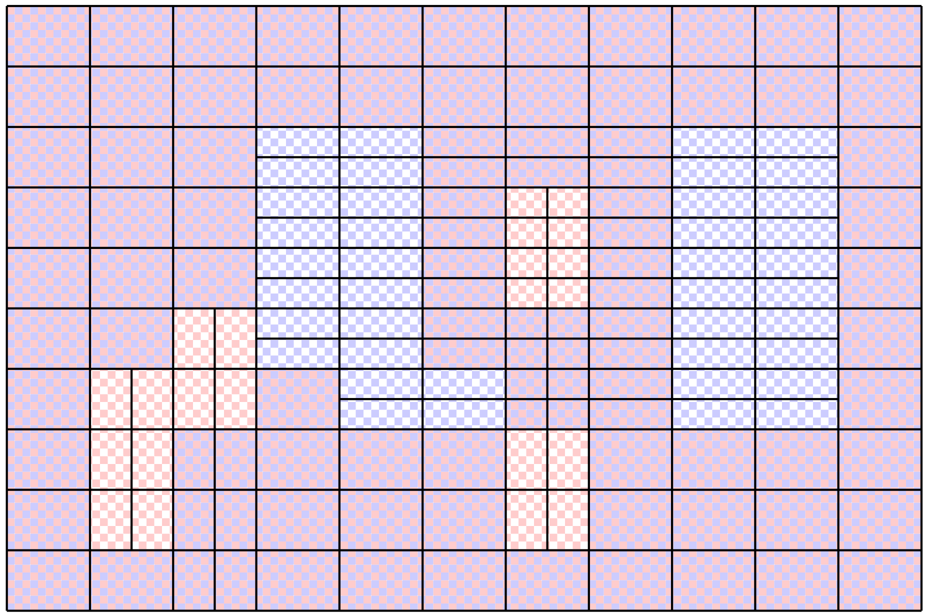


Figure 2: Example of $\mathcal{R}_x$ and $\mathcal{R}_y$ for $\mathbf{d} = (2, 2)$. The region covered by $\bigcup \mathcal{R}_x$ is filled with the red chessboard pattern; that covered by $\bigcup \mathcal{R}_y$ is filled with the blue chess pattern. Their intersection is filled by the blue and red chessboard pattern.

Figure 2 shows the region covered by the rectangles in $\mathcal{R}_x$ and in $\mathcal{R}_y$ for a simple mesh. Note that $\bigcup \mathcal{R}_x$ does not contain the regions near the endpoints of horizontal mesh lines and similarly $\bigcup \mathcal{R}_y$ does not contain the regions near the endpoints of vertical mesh lines.

Lemma 6. *Let $\mathcal{M}$ be an N_2S mesh. If γ is a vertical edge such that $\gamma^\circ \subseteq (\bigcup \mathcal{R}_x(\mathcal{M}))^\circ$ then $\mathcal{M} + \gamma$ is an N_2S-mesh and $\bigcup \mathcal{R}_x(\mathcal{M} + \gamma) \supseteq \bigcup \mathcal{R}_x(\mathcal{M})$. Moreover all the B-splines in $\mathcal{LR}(\mathcal{M} + \gamma)$ either are in $\mathcal{LR}(\mathcal{M})$ or are obtained from B-splines in $\mathcal{LR}(\mathcal{M})$ by the insertion of the abscissa of γ in the horizontal knot vector. Similarly if γ is a horizontal segment.*

Note that the addition of a vertical γ (satisfying the hypothesis) can not reduce the region covered by $\bigcup \mathcal{R}_x$. This means that once $\bigcup \mathcal{R}_x$ is computed it is possible to add many vertical segments well contained in it while preserving the N_2S property. Conversely nothing is said about the behavior of $\bigcup \mathcal{R}_y$ and thus it should be recomputed after each addition of a vertical segment. Similarly for horizontal segments.

Proof. We prove the Lemma only for the addition of a vertical segment $\gamma = \{a\} \times [e, f]$. Consider a B-spline $B = B[\Theta_x, \Theta_y] \in \mathcal{LR}(\mathcal{M})$ whose support is cut by γ in two connected components. This means that B is not a minimal support B-spline with respect to the mesh $\mathcal{M} + \gamma = (\tilde{\mathcal{R}}, \tilde{\sigma})$ and thus there are two B-splines that are obtained from B by inserting a, the abscissa of γ, into Θ_x. Let B' be one of the two B-splines obtained from B. First we prove by contradiction that B' is a minimal support B-spline for $\mathbb{S}(\mathcal{M} + \gamma)$ and then that there is no φ in $\mathcal{LR}(\mathcal{M} + \gamma)$ such that $B' \prec \varphi$. Let Θ'_x be the horizontal knot vector of B'.

Assume that B' is not a minimal support B-spline for $\mathbb{S}(\mathcal{M} + \gamma)$. Then there exists $\varphi \in \mathbb{S}(\mathcal{M} + \gamma)$ that is obtained from B' by knot insertion. Consider first the case of a horizontal knot insertion so that $\varphi = B[\Theta''_x, \Theta_y]$ and let $\bar{\theta}$ be the knot that has been inserted in Θ'_x. Let A be the set of the vertical edges in $\mathcal{E}^v(\mathcal{M} + \gamma)$ that are contained in $(\{\bar{\theta}\} \times \mathbb{R}) \cap \operatorname{supp} B$. The existence of φ implies

$$\forall \alpha \in A,\ \tilde{\sigma}(\alpha) \geq \mu_{\Theta''_x}(\bar{\theta}) > \mu_{\Theta'_x}(\bar{\theta}). \tag{7}$$

From the definition of $\mathcal{M} + \gamma$ it follows

$$\forall \alpha \in A,\ \mu_{\Theta'_x}(\bar{\theta}) = \mu_{\Theta_x}(\bar{\theta}) + c \tag{8}$$

where $c = 1$ if $\bar{\theta} = a$ and 0 otherwise. From the fact that B is in $\mathcal{LR}(\mathcal{M})$ it follows (with the same c)

$$\exists \alpha \in A :\ \tilde{\sigma}(\alpha) = \sigma(\alpha) + c. \tag{9}$$

Equations (7), (8) and (9) are in contradiction.

Consider now the insertion of a vertical knot $\bar{\theta}$ and $\varphi = B[\Theta'_x, \Theta'_y]$. Let A be set of horizontal edges in $\mathcal{E}^h(\mathcal{M} + \gamma)$ contained in $(\mathbb{R} \times \{\bar{\theta}\}) \cap \operatorname{supp} B$. Since $B \in \mathcal{LR}(\mathcal{M})$ and $\gamma^\circ \subseteq (\bigcup \mathcal{R}_x(\mathcal{M}))^\circ$ it follows that for all $\alpha \in A$ such that $\alpha \cap \gamma \neq \emptyset$ it holds

$$\tilde{\sigma}(\alpha) = \mu_{\Theta_y}(\bar{\theta}). \tag{10}$$

On the other hand for $\alpha \subset \operatorname{supp} B'$ it must also hold

$$\tilde{\sigma}(\alpha) \geq \mu_{\Theta'_y}(\bar{\theta}) > \mu_{\Theta_y}(\bar{\theta}). \tag{11}$$

Equations (10) and (11) are in contradiction because there must be at least an α that is contained in $\operatorname{supp} B'$ and intersects γ. We can now conclude that

B' is in $\mathcal{LR}(\mathcal{M}+\gamma)$. Thus the B-spline in $\mathcal{LR}(\mathcal{M}+\gamma)$ whose support intersect $\bigcup \mathcal{R}_x(\mathcal{M})$ have the same vertical knot vector of a B-spline in $\mathcal{LR}(\mathcal{M})$ and thus we can conclude that $\bigcup \mathcal{R}_x(\mathcal{M}+\gamma) \supseteq \bigcup \mathcal{R}_x(\mathcal{M})$.

Assume the existence of $\varphi \in \mathcal{LR}(\mathcal{M}+\gamma)$ such that $B' \prec \varphi$. Then there exists $\varphi' = B[\Xi_x, \Xi_y] \in \mathcal{LR}(\mathcal{M})$ such that $B' \prec \varphi \prec \varphi'$. Let $a_1 < \cdots < a_n$ be the ordinates of the horizontal edges contained in $\operatorname{supp} B$ and intersecting γ. Let $\alpha_i = [a, b_i] \times \{a_i\}$ be the edge intersecting γ in the left endpoint at ordinate a_i. From the hypothesis that $\gamma^\circ \subset (\bigcup \mathcal{R}_x(\mathcal{M}))^\circ$ we conclude that $\alpha_i \subset \bigcup \mathcal{R}_x(\mathcal{M})$ and thus that Ξ_y is compatible with

$$\Big(\underbrace{a_1, \ldots, a_1}_{\sigma(\alpha_1) \text{ times}}, \ldots, \underbrace{a_n, \ldots, a_n}_{\sigma(\alpha_n) \text{ times}}\Big).$$

Moreover from $B' \prec \varphi'$ it follows that $\Xi_y = \Theta_y$. Since $B' \prec \varphi \prec \varphi'$ and all have the same vertical knot vector we conclude that $\varphi = B'$. This proves that $\mathcal{M}+\gamma$ is a $\mathrm{N_2S}$ mesh. □

In Lemma 6 we addressed the preservation of the $\mathrm{N_2S}$ property for the addition of edges that are well contained in $\mathcal{R}_x$ or $\mathcal{R}_y$. This does not cover the limit case, when the added edges intersect the boundary of the refinable region. The problem is that by adding a segment on the boundary of $\mathcal{R}_x$ it is possible to obtain a new B-spline B whose support does not intersect $(\bigcup \mathcal{R}_x)^\circ$. On these B-splines we have no control and they can be further refined and destroy the $\mathrm{N_2S}$ property. We provide sufficient condition for this case in Lemma 7. In Figure 3 there are some examples of vertical segments whose addition preserve the $\mathrm{N_2S}$ property because they satisfy the hypothesis of Lemma 6 (left) or Lemma 7 (right).

Lemma 7. *Let $\gamma = \{a\} \times [e, f]$ be a vertical segment contained in $\bigcup \mathcal{R}_x(\mathcal{M})$. Let $\eta_1, \ldots, \eta_m$ be the maximal (with respect to inclusion) vertical segments contained in $\gamma \cap \partial \bigcup \mathcal{R}_x(\mathcal{M})$. Finally let $\rho_1, \ldots, \rho_n$ be the horizontal edges in $\mathcal{E}^h(\mathcal{M})$ that intersect one of the η_i and are not contained in $\bigcup \mathcal{R}_x(\mathcal{M})$ (see the picture below where $\bigcup \mathcal{R}_x(\mathcal{M})$ is filled with the chessboard pattern). If for each ρ_i there exists $\hat\rho_i \in \mathcal{E}^h(\mathcal{M})$ prolonging ρ_i and contained in $\bigcup \mathcal{R}_x(\mathcal{M})$ such that*

$$\sigma(\hat\rho_i) \geq \sigma(\rho_i)$$

then $\mathcal{M}+\gamma$ is an N_2S-mesh, $\bigcup \mathcal{R}_x(\mathcal{M}+\gamma) \supseteq \bigcup \mathcal{R}_x(\mathcal{M})$ and all B-splines in $\mathcal{LR}(\mathcal{M}+\gamma) \setminus \mathcal{LR}(\mathcal{M})$ are obtained from a B-splines in $\mathcal{LR}(\mathcal{M}) \setminus \mathcal{LR}(\mathcal{M}+\gamma)$

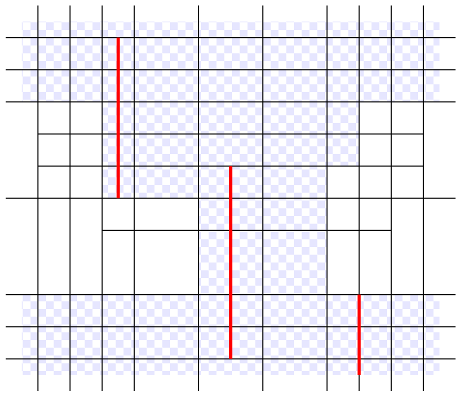
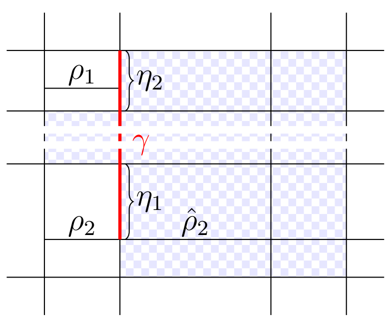


Figure 3: In the meshes above the region covered by $\bigcup \mathcal{R}_x$ is filled with the blue chessboard pattern. The addition of the thick red segments to the mesh preserve the N_2S property. Those drawn on the left mesh satisfy the hypothesis of Lemma 6, those on the right mesh satisfy the hypothesis of Lemma 7. Degree is $(2, 2)$.

by inserting the abscissa of γ *in the horizontal knot vector. Similarly if* γ *is a horizontal segment.*



Proof. The proof follows the pattern of the proof of Lemma 6. First we prove that if $B = B[\Theta_x, \Theta_y] \in \mathcal{LR}(\mathcal{M})$ is split by the insertion of γ then any obtained B-spline $B' = B[\Theta'_x, \Theta_y]$ is in $\mathcal{LR}(\mathcal{M} + \gamma)$.

Assuming that there is $\varphi = B[\Theta''_x, \Theta_y]$ in $\mathbb{S}(\mathcal{M} + \gamma)$ obtained from B' by inserting an horizontal knot we reach the same contradiction as in Lemma 6. Now assume the existence of $\varphi = B[\Theta'_x, \Theta'_y]$ in $\mathbb{S}(\mathcal{M} + \gamma)$ obtained from B' by inserting $\bar{\theta}$ in Θ_y. Let A be the set of the edges in $\mathcal{E}^h(\mathcal{M} + \gamma)$ contained in $\mathbb{R} \times \{\bar{\theta}\} \cap \operatorname{supp} B'$. Then for all α in A it holds

$$\tilde{\sigma}(\alpha) = \sigma(\alpha) \geq \mu_{\Theta'_y}(\bar{\theta}) > \mu_{\Theta_y}(\bar{\theta}). \tag{12}$$

Let $\hat{\alpha}$ be an edge in A that intersect γ. If it is possible to choose $\hat{\alpha} \subset \bigcup \mathcal{R}_x$ then we reach a contradiction as in Lemma 6. Otherwise there must be an index i such that $\hat{\alpha} = \rho_i$. Then by the additional hypothesis it follows

$$\tilde{\sigma}(\hat{\alpha}) = \sigma(\rho_i) \leq \sigma(\hat{\rho}_i) = \mu_{\Theta_y}(\bar{\theta}). \tag{13}$$

Equations (12) and (13) contradict. Thus B' is in $\mathcal{LR}(\mathcal{M} + \gamma)$ and as in Lemma 6 we conclude that $\bigcup \mathcal{R}_x(\mathcal{M} + \gamma) \supseteq \bigcup \mathcal{R}_x(\mathcal{M})$.

Let $\varphi \in \mathcal{LR}(\mathcal{M} + \gamma)$ be such that $B' \prec \varphi$. Then there exists $\varphi' = B[\Xi_x, \Xi_y] \in \mathcal{LR}(\mathcal{M})$ with $B' \prec \varphi \prec \varphi'$.

Let $a_1 < \cdots < a_n$ be the ordinates of the horizontal edges contained in $\operatorname{supp} B$ and intersecting γ. Let $\alpha_i = [a, b_i] \times \{a_i\}$ be the edge intersecting γ in the left endpoint at ordinate a_i. Note that if α_i is not contained in $\bigcup \mathcal{R}_x(\mathcal{M})$ then it must be one of the ρ_j. Call it ρ_{j_i}. Let

$$\hat{\alpha}_i = \begin{cases} \alpha_i & \text{if } \alpha_i \subset \bigcup \mathcal{R}_x(\mathcal{M}), \\ \hat{\rho}_{j_i} & \text{otherwise.} \end{cases} \tag{14}$$

Thus $\operatorname{supp} \varphi'$ contains all of the $\hat{\alpha}_i$ and since they are contained in $\bigcup \mathcal{R}_x(\mathcal{M})$ its knot vector Ξ_y is compatible with

$$\Big(\underbrace{a_1, \ldots, a_1}_{\sigma(\hat{\alpha}_1) \text{ times}}, \ldots, \underbrace{a_n, \ldots, a_n}_{\sigma(\hat{\alpha}_n) \text{ times}}\Big).$$

This and the fact that $B' \prec \varphi'$ imply $\Xi_y = \Theta_y$. Since $B' \prec \varphi \prec \varphi'$ and all have the same vertical knot vector we conclude that $\varphi = B'$. This proves that $\mathcal{M} + \gamma$ is a N_2S mesh. □

4 Hierarchical box meshes

In this section we introduce a hierarchical construction of box meshes. By hierarchical we mean that it starts from a sequence of box meshes associated with nested tensor-product B-spline spaces (tensor meshes). After the definition we describe sufficient conditions for the N_2S property.

4.1 Definition

Let $\mathbb{V}^0 \subset \ldots \subset \mathbb{V}^m$ be a sequence of nested tensor-product B-spline spaces having the same degree $\mathbf{d}$ and defined on the same domain Ω. Let $\mathbb{B}^\ell$ be the canonical basis of the space $\mathbb{V}^\ell$. Each space $\mathbb{V}^\ell$ is defined by a pair of knot vectors $\mathbf{\Theta}^\ell = (\Theta_x^\ell, \Theta_y^\ell)$ whose components are

$$\Theta_x^\ell = (\theta_{x,1}^\ell, \ldots, \theta_{x,n_x^\ell}^\ell),$$
$$\Theta_y^\ell = (\theta_{y,1}^\ell, \ldots, \theta_{y,n_y^\ell}^\ell).$$

In this construction we assume that at each step only one of the two components is refined: either Θ_x^ℓ or Θ_y^ℓ. We say that the refinement at step ℓ is horizontal if $\Theta_x^\ell \neq \Theta_x^{\ell-1}$ and $\Theta_y^\ell = \Theta_y^{\ell-1}$. Similarly we say that it is vertical if $\Theta_x^\ell = \Theta_x^{\ell-1}$ and $\Theta_y^\ell \neq \Theta_y^{\ell-1}$.

Let $\Omega^0 = \Omega \supseteq \ldots \supseteq \Omega^m$ a corresponding sequence of nested domains such that for each level ℓ the domain Ω^ℓ is a union of Bézier elements for $\mathbb{V}^\ell$. That means

$$\Omega^\ell = \bigcup_{(s,t)\in I^\ell} [\theta_{x,s}^\ell, \theta_{x,s+1}^\ell] \times [\theta_{y,t}^\ell, \theta_{y,t+1}^\ell]$$

where I^ℓ is a subset of $\{(i,j) : i = 1, \ldots, n_x^\ell - 1,\, j = 1, \ldots, n_y^\ell - 1\}$. Let $\mathcal{R}^\ell$ be the set of the elements from level ℓ contained in Ω^ℓ, that means

$$\mathcal{R}^\ell = \left\{\eta = [\theta_{x,s}^\ell, \theta_{x,s+1}^\ell] \times [\theta_{y,t}^\ell, \theta_{y,t+1}^\ell] : (s,t) \in I^\ell,\, \eta^\circ \neq \emptyset\right\}.$$

Definition 8. Given a sequence of levels $0, \ldots, m$ as above, the associated *hierarchical LR mesh* is $\mathcal{H} = (\mathcal{R}, \sigma)$ with

$$\mathcal{R} = \bigcup_{\ell=0}^{m} \left\{\eta = \overline{\beta \setminus \Omega^{\ell+1}} : \beta \in \mathcal{R}^\ell,\, \eta \neq \emptyset\right\} \tag{15}$$

and σ defined in terms of the knot multiplicities as follows

$$\sigma(\alpha) = \begin{cases} \mu_{\Theta_x^\ell}(a) & \text{if } \alpha = \{a\} \times [e,f] \in \mathcal{E}^v \\ \mu_{\Theta_y^\ell}(e) & \text{if } \alpha = [a,b] \times \{e\} \in \mathcal{E}^h \end{cases} \tag{16}$$

where ℓ is the biggest index such that $\Omega^\ell \supset \alpha$.

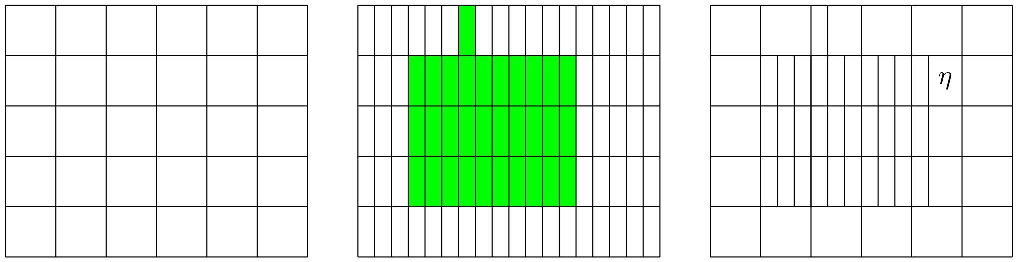


Figure 4: A sample hierarchical mesh with two levels. The element with label η is not an element of any tensor mesh associated to the levels.

Note that all the elements of $\mathcal{R}$ are rectangles because at each step only one direction is refined. Note also that the elements in $\mathcal{R}$ are not necessarily Bezier elements for one of the spaces $\mathbb{V}^0, \ldots, \mathbb{V}^m$, see Figure 4 for a counterexample.

Hierarchical LR meshes can be constructed by iteratively adding segments to a tensor mesh. Let $\mathcal{M}^0$ be the box mesh corresponding to $\mathbb{V}^0$. There are many possible choices of sequences of segments $\gamma_1, \ldots, \gamma_N$ such that

$$\big(\ldots\big((\mathcal{M}^0 + \gamma_1) + \gamma_2\big) + \ldots\big) + \gamma_N = \mathcal{H}.$$

For our purposes it is convenient to define a canonical sequence that we will use in induction proofs. Let $\mathcal{T}^\ell$, $\ell = 0, \ldots, m$ be the hierarchical box meshes associated to the levels $0, \ldots, \ell$. Thus $\mathcal{T}^0$ is the mesh associated to $\mathbb{V}^0$ and $\mathcal{T}^m = \mathcal{H}$. We describe a sequences of additions that construct $\mathcal{T}^\ell$ from $\mathcal{T}^{\ell-1}$. The canonical sequence is then the concatenation of these. Assume that the refinement at step ℓ is horizontal, then we add vertical segments. For each $i = 1, \ldots, n_x^\ell$ we add the connected components of $\Omega^\ell \cap (\{\theta_{x,i}^\ell\} \times \mathbb{R})$ in order of increasing ordinate $\mu_{\Theta_x^\ell}(\theta_{x,i}^\ell) - \mu_{\Theta_x^{\ell-1}}(\theta_{x,i}^\ell)$ times. Similarly for vertical refinement steps.

4.2 Hierarchical N_2S meshes

Restricting to hierarchical box meshes allows us to find sufficient conditions for the N_2S property that can be expressed as constraints on the geometries of $\Omega^0, \ldots \Omega^m$. It is indeed sufficient for the N_2S property that there is enough separation between the boundary of Ω^ℓ and that of $\Omega^{\ell-1}$ in the direction of the refinement. To describe this we introduce a separation distance.

Definition 9. Let $\mathbf{p} = (a, e)$ and $\mathbf{q} = (b, f)$ be points in Ω, then the vertical separation distance between $\mathbf{p}$ and $\mathbf{q}$ relative to level ℓ is a positive integer defined by

$$\operatorname{sep}_y^\ell(\mathbf{p}, \mathbf{q}) = \begin{cases} \#\{j : \theta_{y,j}^\ell \in [e, f]\} & a = b, \\ +\infty & a \neq b. \end{cases}$$

Similarly the horizontal separation is

$$\operatorname{sep}_x^\ell(\mathbf{p}, \mathbf{q}) = \begin{cases} \#\{j : \theta_{x,j}^\ell \in [a, b]\} & e = f, \\ +\infty & e \neq f. \end{cases}$$

Based on the above separations we define the shadow operators $\mathcal{S}^\ell$ that map subsets of Ω to bigger subsets of Ω:

$$\mathcal{S}^\ell A = \begin{cases} \overline{\{\mathbf{p} \in \Omega : \operatorname{sep}_x^\ell(\mathbf{p}, A) \leq d_x\}} & \text{if the refinement at step } \ell+1 \text{ is horizontal,} \\ \overline{\{\mathbf{p} \in \Omega : \operatorname{sep}_y^\ell(\mathbf{p}, A) \leq d_y\}} & \text{if the refinement at step } \ell+1 \text{ is vertical.} \end{cases}$$

See Figure 5 for an example.

Theorem 10. *If the domains associated to the levels are such that $\Omega^\ell \supseteq \mathcal{S}^\ell \Omega^{\ell+1}$, $\ell = 0, \ldots, m-1$ then $\mathcal{H}$ has the N_2S property.*

Proof. We prove the Lemma by induction: we assume that

- $\mathcal{T}^{\ell-2}$ and $\mathcal{T}^{\ell-1}$ are N_2S-meshes;
- if the refinement at level $\ell-1$ is horizontal then $\Omega^{\ell-1} \subseteq \bigcup \mathcal{R}_x(\mathcal{T}^{\ell-2})$;
- if it is vertical $\Omega^{\ell-1} \subseteq \bigcup \mathcal{R}_y(\mathcal{T}^{\ell-2})$;

and prove that

- $\mathcal{T}^\ell$ is an N_2S-mesh
- $\Omega^\ell \subseteq \bigcup \mathcal{R}_x(\mathcal{T}^{\ell-1})$ if the refinement at step ℓ is horizontal;
- $\Omega^\ell \subseteq \bigcup \mathcal{R}_y(\mathcal{T}^{\ell-1})$ if the refinement at step ℓ is vertical.

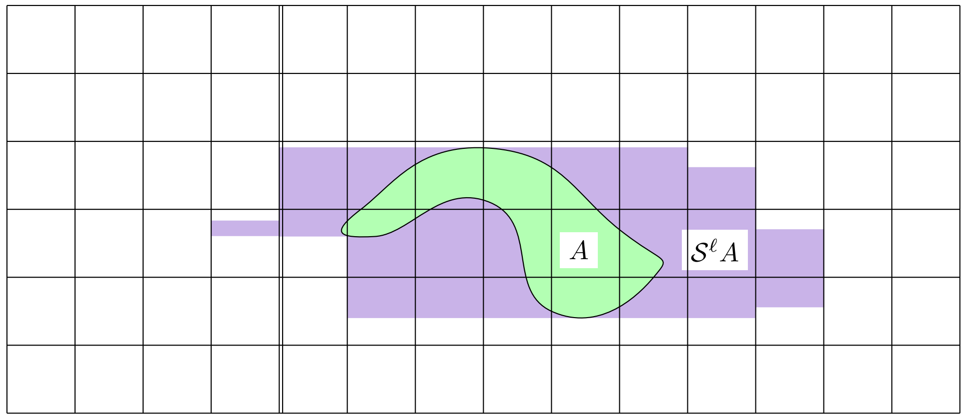


Figure 5: We represent the shadow $\mathcal{S}^\ell A$ (in purple) of a set A (in green) The degree is $(2,2)$, the refinement at step $\ell+1$ is horizontal and the horizontal knot vector of the tensor B-spline space $\mathbb{V}^\ell$ contains a knot with multiplicity 2 that corresponds to the double line in the picture.

Without loss of generality we can assume that the refinement at step ℓ is horizontal, i.e. it corresponds to the addition of vertical segments.

First we prove that $\Omega^\ell \subseteq \bigcup \mathcal{R}_x(\mathcal{T}^{\ell-1})$. There are two cases: either the refinement at step $\ell-1$ is horizontal and thus the above holds as stated in Lemma 6 and Lemma 7 or the refinement at step $\ell-1$ is vertical. In the second case we prove that all elements $\eta = [a,b] \times [e,f]$ of $\mathcal{R}(\mathcal{T}^{\ell-1})$ that intersect Ω^ℓ are in $\mathcal{R}_x(\mathcal{T}^{\ell-1})$. Let $p = (p_x, p_y)$ be a point in $\eta^\circ \cap \Omega^\ell$ and $\alpha_1, \ldots, \alpha_k$ be the vertical edges in $\mathcal{E}^v(\mathcal{T}^{\ell-1})$ that intersect $\mathbb{R} \times \{p_y\} \cap \Omega^{\ell-1}$, see Figure 6.

Let $a_1, \ldots, a_k$ be their abscissa and $\mathscr{X}$ be the knot vector

$$\mathscr{X} = \big(\underbrace{a_1, \ldots, a_1}_{\mu_{\Theta_x^{\ell-1}}(\alpha_1)}, \underbrace{a_2, \ldots, a_2}_{\mu_{\Theta_x^{\ell-1}}(\alpha_2)}, \ldots, \underbrace{a_k, \ldots, a_k}_{\mu_{\Theta_x^{\ell-1}}(\alpha_k)} \big). \tag{17}$$

Since all α_i are contained in $\Omega^{\ell-1} \subseteq \bigcup \mathcal{R}_y(\mathcal{T}^{\ell-2})$ it follows that for all $B[\Xi_x, \Xi_y] \in \mathcal{LR}(\mathcal{T}^{\ell-1})$ such that $\operatorname{supp} B[\Xi_x, \Xi_y] \supseteq \eta$, Ξ_x is compatible with

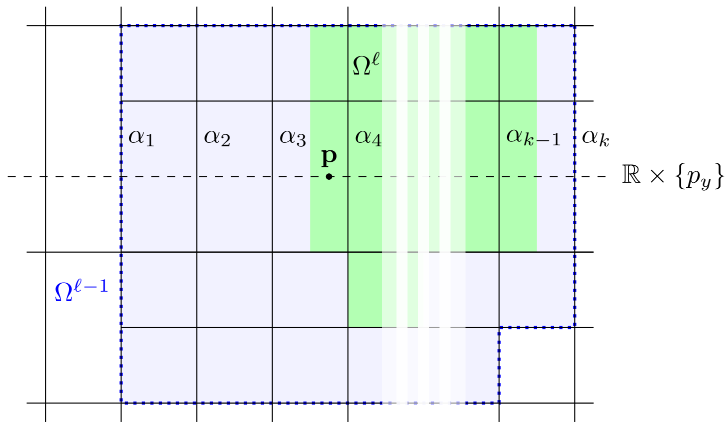


Figure 6: Notation used in the proof of Theorem 10.

$\mathscr{Z}$. Moreover by the definition of the shadow operator it holds

$$\begin{aligned}\sum_{i:a_i<p_x} \mu_{\Theta_x^{\ell-1}}(a_i) \geq d_x + 1,\\ \sum_{i:a_i>p_x} \mu_{\Theta_x^{\ell-1}}(a_i) \geq d_x + 1,\end{aligned} \tag{18}$$

and thus we conclude that

$$\operatorname{supp} B[\Xi_x, \Xi_y] \subset [a_1, a_k] \times \mathbb{R}.$$

Since all edges in $\mathcal{R}^h(\mathcal{M}^{\ell-1})$ contained in $[a_1, a_k] \times \{e\} \cap \Omega^{\ell-1}$ have multiplicity $\mu_{\Theta^{\ell-1}}(e)$ it follows that e must be a knot in Ξ_y with maximum multiplicity: $\mu_{\Xi_y}(e) < \mu_{\Theta^{\ell-1}}(e)$ implies that e is either the first or the last knot. Similarly for f and thus Ξ_y is compatible with

$$\big(\underbrace{e, \ldots, e}_{\mu_{\Theta_y^{\ell-1}}(e)}, \underbrace{f, \ldots, f}_{\mu_{\Theta_x^{\ell-1}}(f)}\big).$$

Since $B[\Xi_x, \Xi_y]$ is arbitrary we conclude that η is contained in $\bigcup \mathcal{R}_x(\mathcal{T}^{\ell-1})$. Since also η is arbitrary we conclude that $\Omega^\ell \subset \bigcup \mathcal{R}_x(\mathcal{T}^{\ell-1})$.

Recall that $\mathcal{T}^\ell$ can be obtained from $\mathcal{T}^{\ell-1}$ by a sequence of additions of segments. Let $\gamma_1, \ldots, \gamma_N$ be the segments described in Subsection 4.1 whose addition to $\mathcal{T}^{\ell-1}$ produces $\mathcal{T}^\ell$. Then the γ_i are contained in $\Omega^\ell \subseteq \mathcal{R}_x(\mathcal{T}^{\ell-1})$. The addition of the γ_i such that ${\gamma_i}^\circ \subset (\bigcup \mathcal{R}_x(\mathcal{T}^{\ell-1}))^\circ$ preserves the N_2S property according to Lemma 6. The others must satisfy the hypothesis of Lemma 7 because of the hierarchical construction. That is: γ can not intercept pairs of aligned connected edges (δ^i, δ^e) with $\delta^i \subset \bigcup \mathcal{R}_x(\mathcal{T}^{\ell-1})$, $\delta^e \not\subset \bigcup \mathcal{R}_x(\mathcal{T}^{\ell-1})$ and $\sigma(\delta^e) > \sigma(\delta^i)$. So the induction is proved. □

5 Completeness

We are also interested in the *completeness* of the provided space, i.e. whether $\mathbb{LR}(\mathcal{H})$ equals $\mathbb{S}(\mathcal{H})$ or not. Describing which refinements preserve completeness was one of the themes of Dokken et al. (2013) and was pursued using homology based techniques. We restrict our attention to hierarchical LR meshes with the N_2S property and we prove that if the Ω^ℓ are "thick enough" in the direction orthogonal to the refinement then the resulting space is $\mathbb{LR}(\mathcal{H}) = \mathbb{S}(\mathcal{H})$. This is made precise in the following result.

Theorem 11. *Let $\mathcal{H}$ be a hierarchical mesh satisfying the hypothesis of Lemma 10: i.e. for all $\ell = 1, \ldots, m$ it holds*

$$\Omega^\ell \supseteq \mathcal{S}^\ell \Omega^{\ell+1}.$$

If for all horizontal refinement steps ℓ and all connected component $A = \{a\} \times [e, f]$ of $\{a\} \times \mathbb{R} \cap \Omega^\ell$ it holds

$$\operatorname{sep}_y^\ell((a, e), (a, f)) \geq d_y + 2, \tag{19}$$

and similarly for vertical refinement steps and all connected component $A = [a, b] \times \{e\}$ of $\mathbb{R} \times \{e\} \cap \Omega^\ell$ it holds

$$\operatorname{sep}_y^\ell((a, e), (b, e)) \geq d_x + 2, \tag{20}$$

then $\mathbb{LR}(\mathcal{H}) = \mathbb{S}(\mathcal{H})$. See Figure 7 for a graphical representation of the hypothesis.

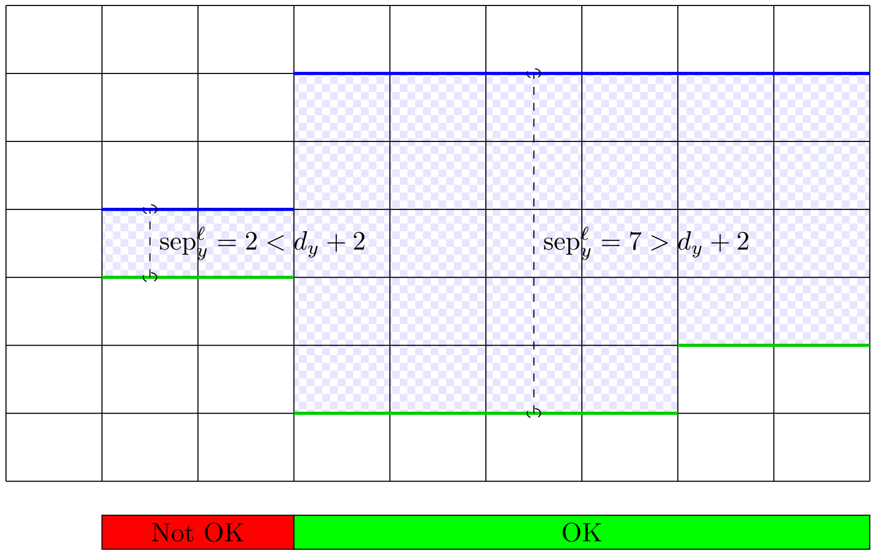


Figure 7: Illustration of the hypothesis of Lemma 11 for a horizontal refinement step ℓ and $d_y = 2$. The region covered by Ω^ℓ is filled with the chessboard pattern. If the separation between the endpoints of the dashed lines (intersections of Ω^ℓ with vertical lines) is greater then $d_y + 2$ then the completeness of $\mathcal{T}^\ell$ is implied from the completeness of $\mathcal{T}^{\ell-1}$.

Proof. As described in Section 4, $\mathcal{H}$ can be obtained by refining the tensor mesh $\mathcal{M}_0$ corresponding to the space $\mathbb{V}^0$:

$$\mathcal{H} = \mathcal{M}_0 + \gamma_1 + \cdots + \gamma_n.$$

Let $\mathcal{M}_i = \mathcal{M}_0 + \gamma_1 + \cdots + \gamma_i$. We know that completeness holds for tensor meshes and thus that

$$\#\mathcal{LR}(\mathcal{M}_0) = \dim \mathbb{S}(\mathcal{M}_0).$$

We prove the induction step:

$$\mathbb{LR}(\mathcal{M}_{i-1}) = \mathbb{S}(\mathcal{M}_{i-1}) \qquad \Rightarrow \qquad \mathbb{LR}(\mathcal{M}_i) = \mathbb{S}(\mathcal{M}_i).$$

Assume that $\mathcal{M}_i$ is finer than $\mathcal{T}^{\ell-1}$ and coarser than $\mathcal{T}^\ell$ and that refinement at step ℓ is horizontal. Then γ_i is vertical segment contained in $\bigcup \mathcal{R}_x(\mathcal{M}_{i-1})$ and a connected component of $\Omega^\ell \cap \{a\} \times \mathbb{R}$. Let $e_1, \ldots, e_k$ be the ordinates of the intersections of γ_i with horizontal edges in $\mathcal{E}^h(\mathcal{M}_{i-1})$, $\mathbf{p}_i = (a, e_1)$ and $\mathbf{q}_i = (a, e_k)$ be the endpoints of γ_i and

$$\Xi = \big(\underbrace{e_1, \dots, e_1}_{\mu_{\Theta_y^{\ell-1}}(e_1)}, \underbrace{e_2, \dots, e_2}_{\mu_{\Theta_y^{\ell-1}}(e_2)}, \dots, \underbrace{e_k, \dots, e_k}_{\mu_{\Theta_y^{\ell-1}}(e_k)}\big).$$

From Theorem 10 we know that $\mathcal{M}_i$ is an $\mathrm{N_2S}$ mesh and thus that

$$\#\mathcal{LR}(\mathcal{M}_i) = \dim \mathbb{LR}(\mathcal{M}_i) \le \dim \mathbb{S}(\mathcal{M}_i).$$

The thesis is implied by $\Delta_{\mathcal{LR}} \ge \Delta_{\mathbb{S}}$ where

$$\Delta_{\mathcal{LR}} = \#\mathcal{LR}(\mathcal{M}_i) - \#\mathcal{LR}(\mathcal{M}_{i-1})$$

and

$$\Delta_{\mathbb{S}} = \dim \mathbb{S}(\mathcal{M}_i) - \dim \mathbb{S}(\mathcal{M}_{i-1}).$$

Reasoning as in the proof of Theorem 10, we deduce that γ_i satisfies the hypotheses of Lemma 6 or Lemma 7. Thus Ξ is compatible with all the vertical knot vectors $\mathscr{Z}_y$ of the B-splines $B[\mathscr{Z}_x, \mathscr{Z}_y] \in \mathcal{LR}(\mathcal{M}_{i-1})$ such that $(\operatorname{supp} B[\mathscr{Z}_x, \mathscr{Z}_y]) \cap \gamma_i \ne \emptyset$. This means that all the $\mathscr{Z}_y$ as above are composed of $d_y + 2$ consecutive knots of Ξ. Thus all linear dependencies in $\mathcal{LR}(\mathcal{M}_i) \setminus \mathcal{LR}(\mathcal{M}_{i-1})$ can be reduced to linear dependencies between B-splines having the same vertical knot vector. Since $\mathcal{M}_i$ is an $\mathrm{N_2S}$ mesh, it follows (Bressan, 2013, Lemma 3.4) that for each point $\mathbf{v} = (v_x, v_y)$ and integers $s \in \{1, \dots, d_x + 1\}$, $t \in \{1, \dots, d_y + 1\}$ there exists $B[\mathscr{Z}_x, \mathscr{Z}_y] \in \mathcal{LR}(\mathcal{M}_{i-1})$ such that

$$\#\{\zeta \in \mathscr{Z}_x : \zeta \le v_x\} = s, \qquad \#\{\zeta \in \mathscr{Z}_y : \zeta \le v_y\} = t.$$

Thus there exists at least a B-spline in $\mathcal{LR}(\mathcal{M}_{i-1})$ that is refined by the addition of γ_i for each knot vectors composed of $d_y + 2$ consecutive knots of Ξ and thus

$$\Delta_{\mathcal{LR}} \ge \sum_{i=1}^{k} \mu_{\Theta_y^{\ell-1}}(e_i) - d_y - 1 = \operatorname{sep}_y^{\ell-1}(\mathbf{p}_i, \mathbf{q}_i) - d_y - 1.$$

Dokken et al. (2013, Theorem 5.5) provide a formula for $\dim \mathbb{S}(\mathcal{M}_i)$. The formula is based on previous research from Mourrain (2014). From the formula it follows

$$\Delta_{\mathbb{S}} = \operatorname{sep}_y^{\ell}(\mathbf{p}_i, \mathbf{q}_i) - d_y - 1 + h_0(\mathcal{M}_i) - h_0(\mathcal{M}_{i-1}) - h_1(\mathcal{M}_i) + h_1(\mathcal{M}_{i-1}) \quad (21)$$

where h_0, h_1 are two non-negative functions of the mesh that can be computed with homological techniques. According to Dokken et al. (2013, Note 1 after Theorem 5.5), $h_0(\mathcal{M}_i) = 0$ if $\mathbb{S}(\mathcal{M}_i) \neq \{0\}$ and this is our case. The homological term $h_1(\mathcal{M}_0)$ is 0 (Note 2). Moreover $h_1(\mathcal{M}_i)$ is non-increasing with respect to i because of (19), (20) and (Dokken et al., 2013, Note 3). Thus all $h_0(\mathcal{M}_{i-1})$, $h_0(\mathcal{M}_i)$, $h_1(\mathcal{M}_{i-1})$ and $h_1(\mathcal{M}_i)$ are 0 and

$$\Delta_{\mathcal{LR}} \geq \operatorname{sep}_y^\ell(\mathbf{p}_i, \mathbf{q}_i) - d_y - 1 = \Delta_{\mathbb{S}}. \tag{22}$$

□

6 Construction

In this subsection we present a construction for hierarchical box meshes that guarantees both the N_2S property and completeness. We assume that the spaces $\mathbb{V}^0, \dots, \mathbb{V}^m$ are fixed and that a minimum refinement level is specified for some regions.

The input of our construction is a sequence of $\omega^1, \dots, \omega^m$ of subsets of $\Omega = [0,1]^2$ and the output is a mesh such that all basis functions that are active on a point in ω^ℓ are refinements of basis functions from $\mathbb{B}^\ell$. The ω^ℓ do not need to be nested and can be empty. For example they can be a discrete set of points, a curve or an region or a union of those.

From $\omega_1, \dots, \omega_m$ we construct the domains $\Omega^1 \supseteq \dots \supseteq \Omega^m$ in order of decreasing m. First we define the auxiliary sets $\tilde{\omega}^\ell$ for $\ell = 1, \dots, m$:

$$\tilde{\omega}^\ell = \bigcup \{\operatorname{supp} B : B \in \mathbb{B}^\ell, \operatorname{supp} B \cap \omega^\ell \neq \emptyset\}.$$

Set Ω^m to $\tilde{\omega}^m$. Then $\Omega^{\ell-1}$ $i = m-1, \dots, 1$ is defined to be

$$\Omega^{\ell-1} = \tilde{\omega}^{\ell-1} \cup \mathcal{S}^{\ell-1}\Omega^\ell. \tag{23}$$

By construction the domains associated to the levels satisfy Lemma 10 and Lemma 11 and thus $\mathcal{H}$ is an N_2S-mesh for which $\mathbb{LR}(\mathcal{H}) = \mathbb{S}(\mathcal{H})$.

We conclude this section with some examples of the meshes constructed with our method in case of dyadically refined knot vectors. Let

$$\Theta_x^\ell = [\,\underbrace{0, \dots, 0}_{d_x+1 \text{ times}}, \dots \frac{k}{2^{\lceil \ell/2 \rceil}} \dots, \underbrace{1, \dots, 1}_{d_x+1 \text{ times}}\,], \qquad 0 < k < 2^{\lceil \ell/2 \rceil}$$

$$\Theta_y^\ell = [\,\underbrace{0, \dots, 0}_{d_y+1 \text{ times}}, \dots \frac{k}{2^{\lfloor \ell/2 \rfloor}} \dots, \underbrace{1, \dots, 1}_{d_y+1 \text{ times}}\,], \qquad 0 < k < 2^{\lfloor \ell/2 \rfloor}.$$

We show the meshes constructed from the input regions $\omega^1, \ldots, \omega^m$ with

$$\omega^\ell = \begin{cases} G & \ell = m, \\ \emptyset & \text{otherwise} \end{cases}$$

for different choices of G and m.

In Figure 8 the set G is a polygonal chain composed of two segments and m is $4, 6, 8, 10, 12, 14$. In Figure 9 it is a spiral centered in $(0.5, 0.5)$ and m is $4, 6, 8, 10, 12, 14$. In both Figures the degree is $\mathbf{d} = (2, 2)$.

As can be seen in Figure 8 and 9 the refined region follows G closely and does not propagate. This statement can be made more precise. For simplicity we assume that the degree is the same in both coordinate directions $d_x = d_y = d$ and that the maximal level m is even.

The size of an element $\eta = [a, b] \times [e, f] \in \mathcal{R}(\mathcal{H})$ contained in $\Omega^\ell \setminus \Omega^{\ell+1}$ is bounded by

$$\begin{aligned} 2^{-\lceil \ell/2 \rceil - 1} \leq & b - a \leq 2^{-\lceil \ell/2 \rceil} \\ 2^{-\lfloor \ell/2 \rfloor - 1} \leq & f - e \leq 2^{-\lfloor \ell/2 \rfloor}. \end{aligned}$$

The distance of such an element from ω^m can be estimated using a geometric sum. Indeed the distance between ω^m and $\partial\Omega^m$ is in

$$[d 2^{-m/2},\ (d+1) 2^{-m/2+1/2}].$$

The operator $\mathcal{S}^{\ell-1}$ extends the domain Ω^ℓ in one direction by a length contained between $d 2^{-\ell/2+1}$ and $(d+1)2^{-\ell/2+1}$. It follows that the distance between Ω^ℓ and the boundary of $\Omega^{\ell-2}$ (in the case of ℓ even) is contained between $d2^{-\ell/2+1}$ and $(d+1)2^{-\ell/2+3/2}$.

The distance between $\eta \subseteq \Omega^\ell \setminus \Omega^{\ell+1}$ and ω^m can be bounded using a geometric sum and for $\ell < m - 1$ it is contained in

$$[d(2^{-\ell/2} - 2^{-m/2}), 2^{3/2}(d+1)2^{-\ell/2}].$$

So we proved that any point contained in an element of size $\approx 2^{-\ell/2}$ is at a distance $\approx 2^{-\ell/2}$ from ω^m. This means that the obtained meshes are geometrically refined only near the requested regions.

7 Comparison with THB-splines

It is interesting to compare the described approach with the THB-spline approach developed by Giannelli et al. (2012); Mokriš et al. (2014). We do

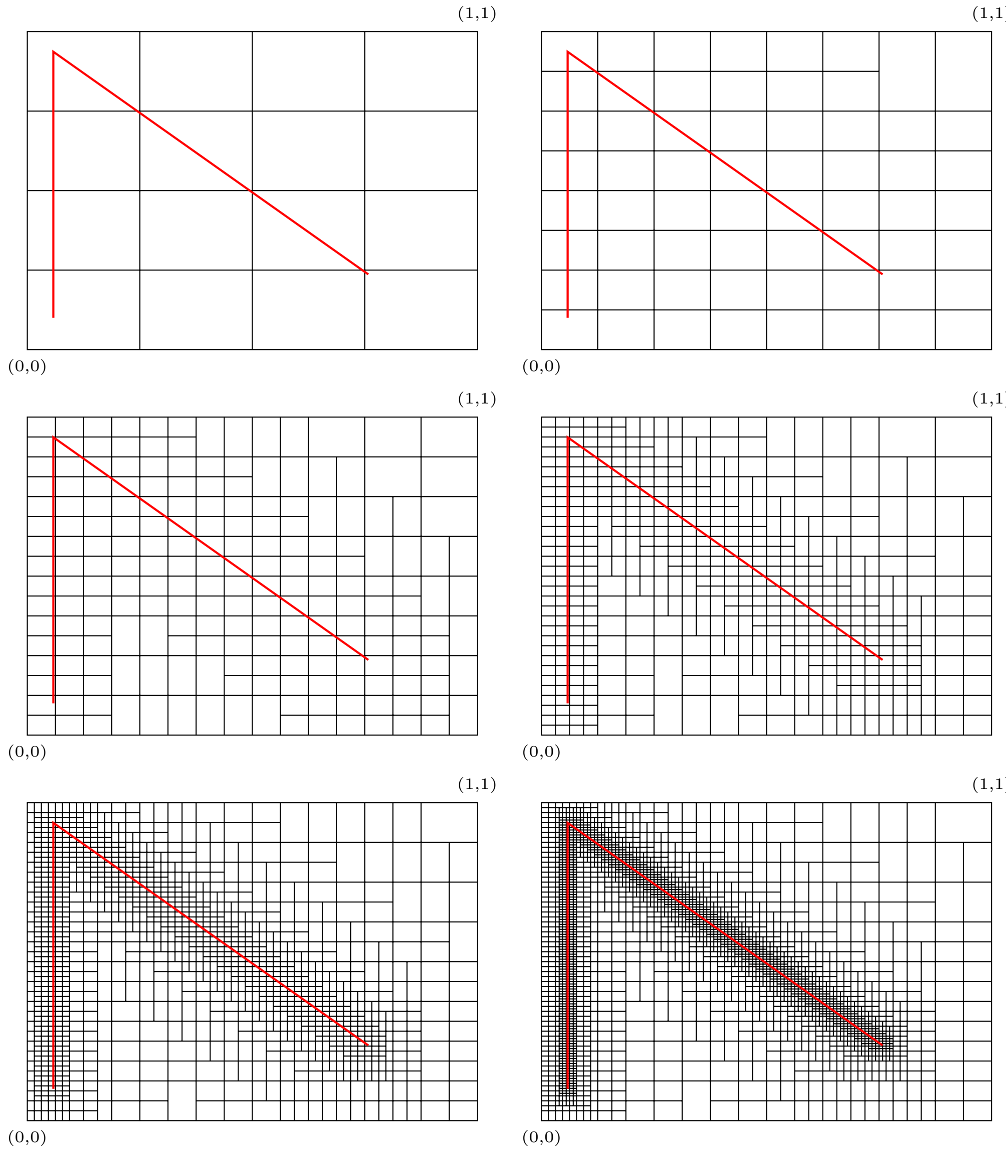


Figure 8: Meshes obtained by setting $\omega^m = \Gamma$ for $m = 4, 6, \ldots, 14$. The degree is $(2, 2)$.

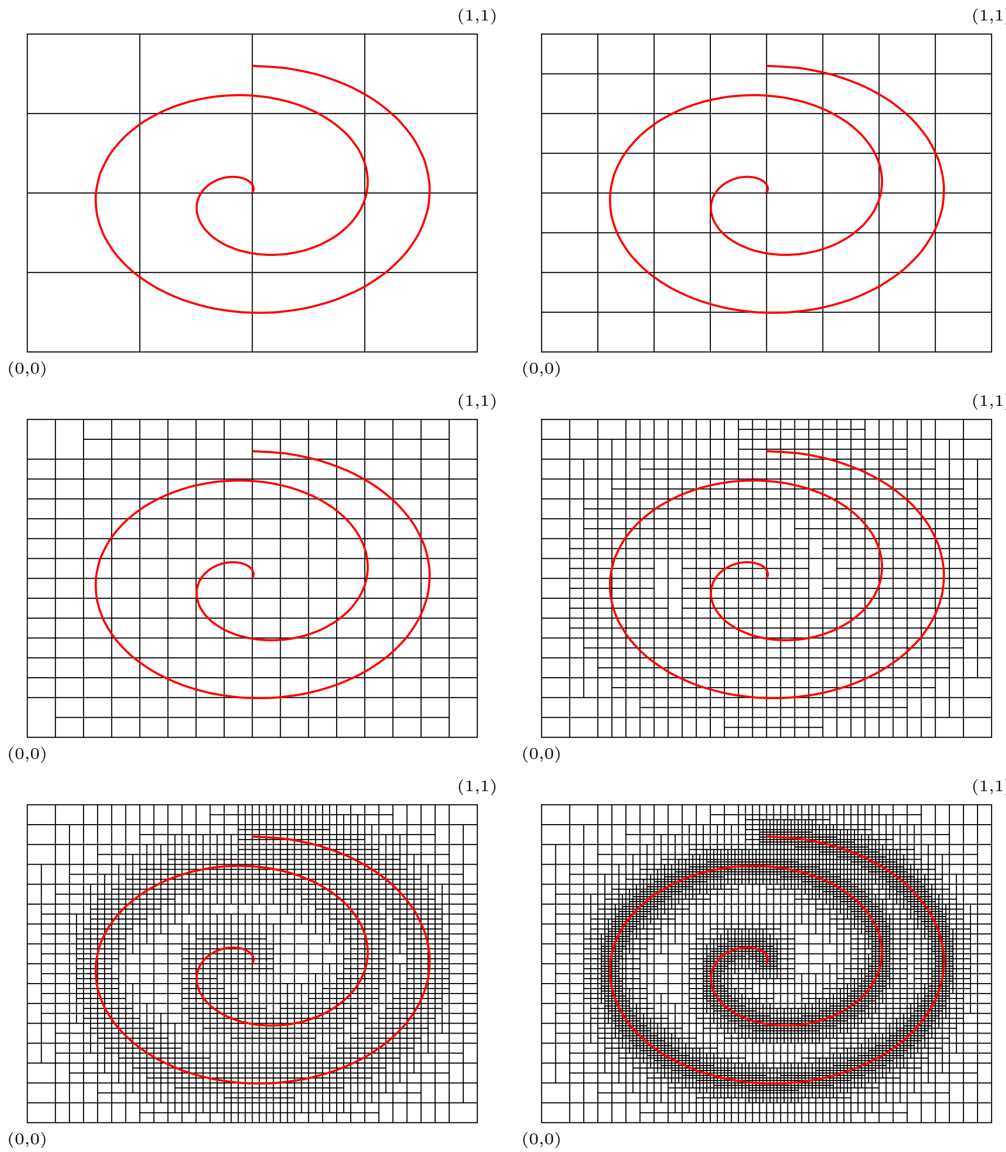


Figure 9: Meshes obtained by setting $\omega^m = \Gamma$ for $m = 4, 6, \ldots, 14$. The degree is $(2, 2)$.

this in the simplified setting in which Ω^ℓ is a union of rectangles of level $\ell - 1$. In this setting we can compare the space $\mathbb{LR}(\mathcal{H})$ to the THB-spline space $\mathbb{TH}(\mathcal{H})$ having the same Bézier elements and defined from the same levels. This means that $\mathbb{LR}(\mathcal{H})$ and $\mathbb{TH}(\mathcal{H})$ are defined by the same sequence of tensor-product spaces $\mathbb{V}^\ell$ and domains Ω^ℓ.

The hierarchical spline construction selects a subset $\mathcal{C}^\ell$ from the tensor-product basis $\mathbb{B}^\ell$ of $\mathbb{V}^\ell$. Precisely

$$\mathcal{C}^\ell = \left\{ B \in \mathbb{B}^\ell : \operatorname{supp} B \subseteq \Omega^\ell \wedge \operatorname{supp} B \not\subseteq \Omega^{\ell+1} \right\}.$$

The set of generators is then $\bigcup_{\ell=0}^{m} \mathcal{C}^\ell$.

The THB-spline approach uses the same selection procedure, but the selected B-splines are *truncated* in order to guarantee that they are a partition of unity and to obtain better locality. In particular each B-spline B selected from level ℓ whose support intersects $\Omega^{\ell+1}$ is replaced by the function $\hat{B}_{\ell+1}$ obtained by expressing B as a linear combination of B-splines from level $\ell+1$ and by setting the coefficients of the B-splines in $\mathcal{C}^{\ell+1}$ to 0. The procedure is then repeated for each lower level $\ell + 2, \ldots, m$. At this point the collection of truncated B-splines is taken as the basis of the space. The truncated functions are always a partition of unity.

The space $\mathbb{TH}(\mathcal{H})$ is a subset of $\mathbb{S}(\mathcal{H})$ and equality is proved for meshes such that for each level ℓ and $B \in \mathcal{C}^\ell$ the intersection $(\operatorname{supp} B)^\circ \cap (\Omega \setminus \Omega^{\ell+1})$ is connected. Sharper results can be obtained for specific degrees and weaker conditions are needed for the decoupled version of the THB-basis that was proposed by Mokriš et al. (2014). This condition is not always satisfied by the type of meshes we are considering as can be seen in Figure 10.

So in selected cases the space we are proposing is bigger than the THB-spline space. Two other advantages are that the basis functions are B-splines and that the basis functions are locally linearly independent.

On the other hand the locality of the refinement with our approach is degree dependent and decreases as the degree increase. This does not happen in the THB-spline setting. Related to this is the fact that THB-splines do not require “alternating” refinement and thus can be refined more locally.

8 Conclusions

We restrict our attention to the subset of LR meshes that have the N_2S-property. We describe two subdomains of Ω where respectively vertical and

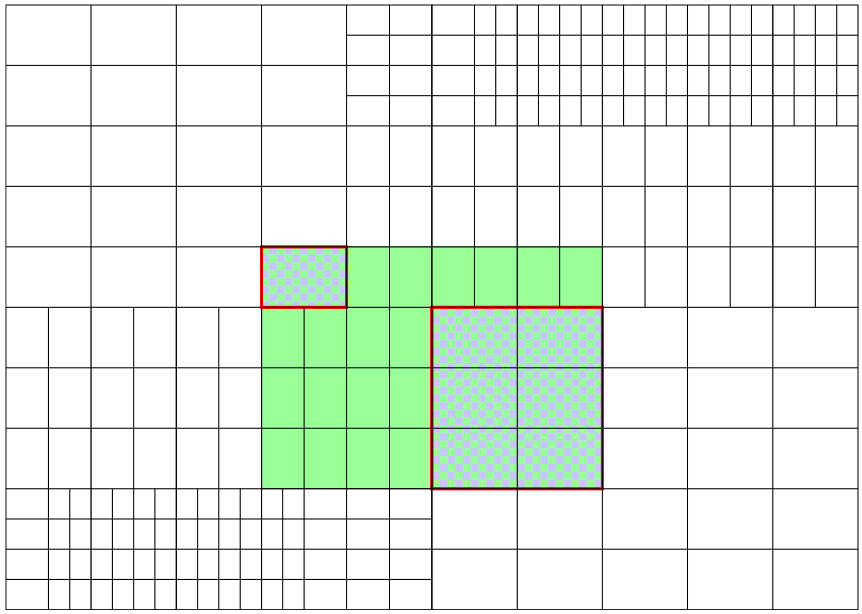


Figure 10: In this example mesh the support of the B-spline B of degree $(3,3)$ from the coarsest level intersect the $\Omega\backslash\Omega^1$ in two connected components highlighted with the chess pattern. Therefore completeness is not guaranteed in the THB, but it is by N_2S. In this example $\dim \mathbb{S} = \dim \mathbb{LR} = 331 > \dim \mathbb{TH} = 328$.

horizontal refinement preserves the N_2S property. Using this knowledge we provide an explicit construction that is based on a hierarchy of tensor spaces and domains. The LR-space associated to the constructed mesh $\mathcal{H}$ has the N_2S property, i.e. it has a basis of locally linearly independent functions. Moreover $\mathbb{LR}(\mathcal{H})$ is the whole space $\mathbb{S}(\mathcal{H})$ of piecewise polynomials associated to the mesh.

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