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Derivatives of Isogeometric Functions on Rational Patches

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Abstract

We consider isogeometric functions and their derivatives. Given a geometry mapping as an n-dimensional NURBS patch in $\mathbb{R}^d$, an isogeometric function is obtained by composing the inverse of the geometry mapping with a NURBS function in the parameter domain. Hence an isogeometric function can be represented by a NURBS patch parametrizing its graph.

Using this approach, we develop a closed form representation of the graph of a partial derivative of an isogeometric function. The derivative can be interpreted as an isogeometric function on the same piecewise rational geometry mapping, hence the space of isogeometric functions is closed under differentiation. All computations are described compactly using homogeneous coordinates. We distinguish the two cases $n = d$ and $n < d$, with a focus on $n = d - 1$ in the latter one.

We use the previously developed representation of the derivatives to derive conditions which guarantee $\mathscr{C}^1$ and $\mathscr{C}^2$ smoothness for isogeometric functions on two types of singularly parametrized planar domains as well as on a part of a spherical cap. It is interesting to note that the presented conditions depend heavily on the representation of the interior of the given geometry mapping.

Keywords: isogeometric analysis, isogeometric function, derivative, NURBS, singular patch, smoothness, embedded manifold, spherical cap

1. Introduction

This work is motivated by isogeometric analysis (IgA), introduced by Hughes et al. (2005). IgA directly uses the NURBS based representation of a CAD model to construct a test/trial function space for numerical simulations. An isogeometric function is the composition of a NURBS function with the inverse of the geometry mapping. In this paper we study the derivatives of such isogeometric functions in more detail.

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In IgA, the space of isogeometric functions is used to discretize a partial differential equation on the physical domain. One approach is to use the isogeometric functions as test/trial functions for a Galerkin discretization of the variational formulation (see e.g. Hughes et al., 2005; Gomez et al., 2008; Cottrell et al., 2009). Another possibility is to use them as a basis for collocation (see Auricchio et al., 2010; Schillinger et al., 2013). Consequently, it is necessary to approximate the derivatives of the test functions or to evaluate them point-wise. Fast and stable algorithms to compute the derivatives are of vital interest. If one is interested in the smoothness or in the regularity of an isogeometric functions, it is necessary to have a representation of its derivatives. As the main contribution of our paper, we provide a simple closed-form representation for the derivatives of the isogeometric functions as parametric patches. We deal with domains represented by n-dimensional patches in $\mathbb{R}^d$, for both $n = d$ and $n < d$, where we especially focus on $n = d - 1$ (like e.g. a surface patch in $\mathbb{R}^3$). The present paper is an extension of the work by Takacs and Jüttler (2013), where we were dealing with planar domains only.

An isogeometric function on a physical domain, defined by a NURBS patch, can be represented as a NURBS patch parametrizing its graph. If the domain is a d-dimensional patch in $\mathbb{R}^d$, then the graph of an isogeometric function is a d-dimensional patch (a hyper-surface) in $\mathbb{R}^{d+1}$. The patch is given in homogeneous coordinates to simplify the notation. Using this approach, we develop a closed form representation of the graphs of the partial derivatives of an isogeometric function. The derivatives can again be interpreted as isogeometric functions on the same piecewise rational geometry mapping. Therefore the space of isogeometric functions on a given domain patch is closed under differentiation.

Using the presented method to compute the derivatives of isogeometric functions we derive conditions that guarantee smoothness when dealing with patches with singular parametrizations. Related results concerning the smoothness of singular rational Bézier patches were achieved by Bohl and Reif (1997); Sederberg et al. (2011). However, our method provides a systematic approach to derive conditions which characterize smoothness of arbitrary order for singular n-dimensional patches in $\mathbb{R}^d$. We present results for two parametrizations of planar domains as well as for a part of a spherical cap to demonstrate the capabilities of the proposed method of smoothness analysis. Patches such as the spherical shell are of importance for applications such as shell analysis (see e.g. Kiendl et al., 2009; Benson et al., 2010). The results concerning the smoothness of isogeometric functions are related to the results by Takacs and Jüttler (2011, 2012) about H^1 and H^2 regularity properties on singular patches.

We consider singularly parametrized domains, since they are particularly useful for the modeling of physical domains of general shape. Higher dimensional NURBS possess a tensor-product structure, hence regular mappings can only describe domains that have a box-structure. When dealing with singularities, the smoothness of the isogeometric functions might be reduced. This may also affect the stability and convergence properties of the numerical schemes applied. Contributions to the theoretical background of IgA include the numerical analysis concerning consistency and stability of the method (Bazilevs et al., 2006; Cottrell et al., 2007; Hughes et al., 2010; Echter and Bischoff, 2010). However, singularly parametrized domains are not covered by these general results and have to be treated separately.

After this short introduction and motivation we given an overview of the remainder of this paper. In Section 2 we introduce the notation of isogeometric functions on rational patches. We first introduce the homogeneous coordinate representation of rational n-dimensional patches in $\mathbb{R}^d$ in Subsection 2.1. In the following Subsection 2.2 we introduce isogeometric function on such patches. We conclude the section with a short introduction to isogeometric analysis. In Section 3 we treat the case of derivatives on patches with $n = d$. We then discuss $\mathscr{C}^1(\bar{\Omega})$ and $\mathscr{C}^2(\bar{\Omega})$ smoothness properties of isogeometric functions on planar domains Ω in Subsection 3.2. We denote by $\mathscr{C}^k(\bar{\Omega})$ the space of functions where the k-th derivatives are continuous in the interior of the domain Ω and can be extended continuously to the boundary $\partial\Omega$. In Section 4 we are extending the results to n-dimensional patches in $\mathbb{R}^d$ with $n < d$. We especially focus on surfaces in $\mathbb{R}^3$ in Subsection 4.2, where we consider as a model case a part of a spherical cap. We conclude the paper with some final remarks in Section 5.

2. Isogeometric functions on rational patches

In the following we introduce rational n-dimensional patches which are embedded into the Euclidean space of dimension d. The patches are given in homogeneous coordinates, which we present in Subsection 2.1. Then we define isogeometric functions on these patches in Subsection 2.2. The motivation of this representation comes from isogeometric analysis, which we briefly recall at the end of this section.

2.1. Rational n-dimensional patches in $\mathbb{R}^d$

Throughout this section we deal with (piecewise) rational parametrizations of patches in $\mathbb{R}^d$. Since we are considering (piecewise) rational functions we can take advantage of the concept of homogeneous coordinates. This is a common approach when dealing with rational functions or NURBS (see e.g. Farin, 1999).

Any Cartesian coordinate vector $\mathbf{r} = [r_1, \ldots, r_d]^T$ in $\mathbb{R}^d$ can be represented in homogeneous coordinates by

$$\tilde{\mathbf{r}} = (\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_d)^T \in \mathbb{R}\backslash\{0\} \times \mathbb{R}^d,$$

where $r_i = \tilde{r}_i/\tilde{r}_0$ for $i \in \{1, \ldots, d\}$. Hence any two points $\mathbf{r}$ and $\mathbf{s}$ are identical if and only if the homogeneous coordinate vectors $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{s}}$ are linearly dependent. More precisely, a point with the Cartesian coordinates $\mathbf{r}$ corresponds to the equivalence class $\tilde{\mathbf{r}}\mathbb{R}$ of points in homogeneous coordinates. Note that homogeneous coordinates are always written with a tilde ($\tilde{\mathbf{r}}$) and Cartesian coordinates without a tilde ($\mathbf{r}$).

A piecewise rational patch in Cartesian coordinates

$$\mathbf{r}(\mathbf{u}) = \left(\frac{\tilde{r}_1(\mathbf{u})}{\tilde{r}_0(\mathbf{u})}, \ldots, \frac{\tilde{r}_d(\mathbf{u})}{\tilde{r}_0(\mathbf{u})}\right)^T \quad \text{for } \mathbf{u} \in \mathbf{B} \subset \mathbb{R}^n$$

can be represented by a piecewise polynomial patch in homogeneous coordinates, with

$$\tilde{\mathbf{r}}(\mathbf{u}) = (\tilde{r}_0(\mathbf{u}), \tilde{r}_1(\mathbf{u}), \ldots, \tilde{r}_d(\mathbf{u}))^T \quad \text{for } \mathbf{u} \in \mathbf{B}.$$

The parameter domain $\mathbf{B}$ of the patch is an n-dimensional open box. Note that $\tilde{r}_0, \tilde{r}_1, \ldots, \tilde{r}_{d+1} \in \mathcal{S}^{\mathbf{p}}$, where $\mathcal{S}^{\mathbf{p}}$ is a space of tensor-product B-splines of degree $\mathbf{p} = (p_1, \ldots, p_n)$ with respect to the parameters $\mathbf{u}$. For details on NURBS and B-splines and their applications in computer aided geometric design we refer to Farin (1999); Prautzsch et al. (2002); Hoschek and Lasser (1993).

We recall the tensor-product B-spline representation of $\tilde{\mathbf{r}}$ with respect to the basis $\{B_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{I}}$ of $\mathcal{S}^{\mathbf{p}}$,

$$\tilde{\mathbf{r}} : \ \mathbf{u} \mapsto (\tilde{r}_0(\mathbf{u}), \ldots, \tilde{r}_d(\mathbf{u}))^T = \sum_{\mathbf{i}\in\mathbb{I}} \tilde{\mathbf{c}}_{\mathbf{i}} B_{\mathbf{i}}(\mathbf{u}), \tag{1}$$

with homogeneous control points $\tilde{\mathbf{c}}_{\mathbf{i}} \in \mathbb{R}^{d+1}$. In Cartesian coordinates this transforms to

$$\mathbf{r} : \ \mathbf{u} \mapsto (r_1(\mathbf{u}), \ldots, r_d(\mathbf{u}))^T = \sum_{\mathbf{i}\in\mathbb{I}} \mathbf{c}_{\mathbf{i}} R_{\mathbf{i}}(\mathbf{u}), \tag{2}$$

with Cartesian control points $\mathbf{c}_{\mathbf{i}} = (\tilde{c}_{\mathbf{i},1}/\tilde{c}_{\mathbf{i},0}, \ldots, \tilde{c}_{\mathbf{i},d}/\tilde{c}_{\mathbf{i},0})^T \in \mathbb{R}^d$, weights $\tilde{c}_{\mathbf{i},0}$ and piecewise rational basis functions

$$R_{\mathbf{i}}(\mathbf{u}) = \frac{\tilde{c}_{\mathbf{i},0} B_{\mathbf{i}}(\mathbf{u})}{\tilde{r}_0(\mathbf{u})}, \quad \text{with} \quad \tilde{r}_0(\mathbf{u}) = \sum_{\mathbf{j}\in\mathbb{I}} \tilde{c}_{\mathbf{j},0} B_{\mathbf{j}}(\mathbf{u}). \tag{3}$$

The (physical) domain $\Omega \subset \mathbb{R}^d$ is given as the image of the geometry mapping $\mathbf{r}$, i.e. $\Omega = \mathbf{r}(\mathbf{B})$. In the following subsection we define isogeometric functions on the domain Ω.

2.2. Isogeometric functions on rational patches

The notion of isogeometric functions as well as the results presented in this paper are motivated by isogeometric analysis, which we recall at the end of this section.

Definition 1. Let $\mathbf{B} \subset \mathbb{R}^n$ be an open box. Given an invertible NURBS patch $\mathbf{r} : \mathbf{B} \to \Omega \subset \mathbb{R}^d$ as in (2) with $\tilde{\mathbf{r}}$ as in (1) such that $\tilde{r}_j \in \mathcal{S}^{\mathbf{p}}$, for $0 \leq j \leq d$. Considering another B-spline $\tilde{r}_{d+1} \in \mathcal{S}^{\mathbf{p}}$, the function $f : \Omega = \mathbf{r}(\mathbf{B}) \to \mathbb{R}$ defined via

$$f(\mathbf{r}(\mathbf{u})) = \frac{\tilde{r}_{d+1}(\mathbf{u})}{\tilde{r}_0(\mathbf{u})}$$

is called an *isogeometric function* on Ω.

By $\mathcal{V}^{\mathbf{p}}(\Omega)$ we denote the space of isogeometric functions corresponding to $\mathcal{S}^{\mathbf{p}}$. By $\mathcal{V}(\Omega)$ we denote the space of isogeometric functions with $\tilde{r}_{d+1} \in \mathcal{P}(\mathbf{B})$, where $\mathcal{P}(\mathbf{B})$ is the space of piecewise polynomials on $\mathbf{B}$.

Figure 1 gives a schematic overview of the functions $\mathbf{r}$, f and $f \circ \mathbf{r}$.

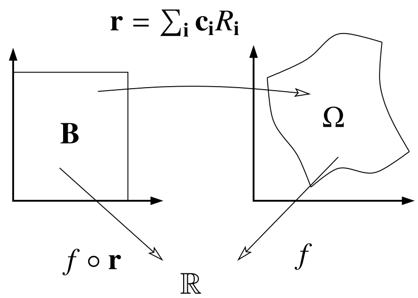

Figure 1: Bivariate geometry mapping $\mathbf{r}$ with parameter domain $\mathbf{B}$, physical domain Ω, isogeometric function f and pullback $f \circ \mathbf{r}$

From Definition 1 we get that any isogeometric function $f \in \mathcal{V}^{\mathbf{p}}$ has a representation in the basis $R_{\mathbf{i}} \circ \mathbf{r}^{-1}$ with coefficients $\tilde{f}_{\mathbf{i}}/\tilde{c}_{\mathbf{i},0}$ where $R_{\mathbf{i}}$ is given as in (3), i.e.

$$f = \left(\frac{\tilde{r}_{d+1}}{\tilde{r}_0}\right) \circ \mathbf{r}^{-1} = \left(\sum_{\mathbf{i}\in\mathbb{I}} \frac{\tilde{f}_{\mathbf{i}}}{\tilde{c}_{\mathbf{i},0}} R_{\mathbf{i}}\right) \circ \mathbf{r}^{-1}.$$

The graph of the function $f \in \mathcal{V}^{\mathbf{p}}$ can be represented by the parametric surface $\tilde{\mathbf{f}}$ in homogeneous coordinates with

$$\tilde{\mathbf{f}}(\mathbf{u}) = (\tilde{r}_0(\mathbf{u}), \tilde{r}_1(\mathbf{u}), \ldots, \tilde{r}_d(\mathbf{u}), \tilde{r}_{d+1}(\mathbf{u}))^T, \tag{4}$$

having the B-spline representation

$$\tilde{\mathbf{f}}(\mathbf{u}) = \sum_{\mathbf{i}\in\mathbb{I}} \begin{pmatrix} \tilde{\mathbf{c}}_{\mathbf{i}} \\ \tilde{f}_{\mathbf{i}} \end{pmatrix} B_{\mathbf{i}}(\mathbf{u}),$$

with homogeneous control points $(\tilde{c}_{\mathbf{i},0}, \ldots, \tilde{c}_{\mathbf{i},d}, \tilde{f}_{\mathbf{i}})^T \in \mathbb{R}^{d+2}$ for $\mathbf{i} \in \mathbb{I}$.

We present two simple examples to give a geometric interpretation of isogeometric functions. The left part of Figure 2 depicts a bivariate geometry mapping $\mathbf{r}$ of a planar domain Ω and the graph surface $\mathbf{f}$ of an isogeometric function f defined on Ω. The right part visualizes an isogeometric function f defined over a planar curve given by $\mathbf{r}$.

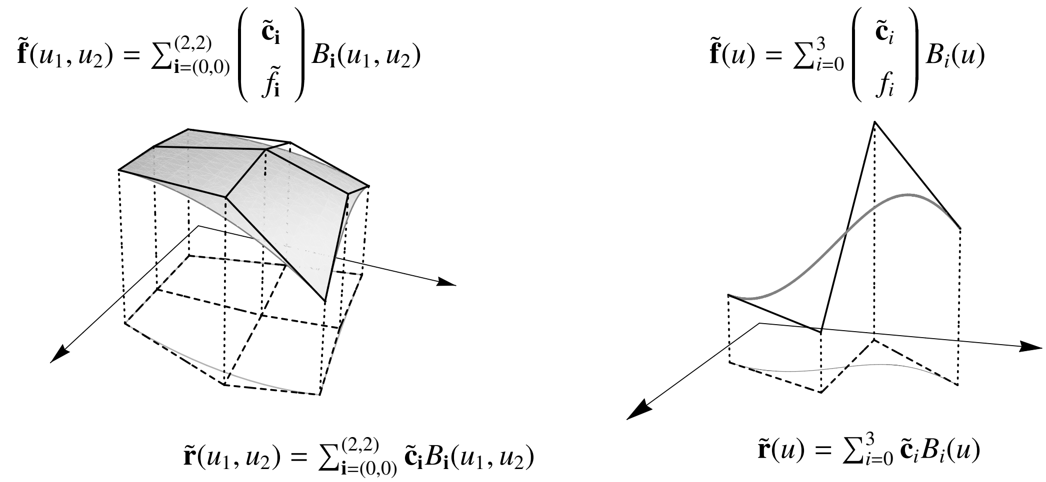

Figure 2: Planar patch **r** (left), planar curve **r** (right) and corresponding graph surfaces **f** with their respective representations in homogeneous coordinates

2.3. Motivation: Isogeometric analysis

The focus of this paper lies in the study of derivatives of isogeometric functions $f \in \mathcal{V}$. Isogeometric analysis, introduced by Hughes et al. (2005), is based on isogeometric functions of NURBS. In that case the geometry mapping as well as the pull back of f are NURBS.

Isogeometric analysis is a numerical solution method for partial differential equations on domains derived from CAD geometries. In isogeometric analysis the finite-dimensional space of test functions $\mathcal{V}^{\mathbf{p}}$ is used to serve as a solution space for the discretized physical problem on the domain $\Omega \subset \mathbb{R}^d$.

In many applications one is interested either in point-wise evaluation or in the exact computation of derivatives of isogeometric functions. In the following we give a closed form representation of the derivatives of isogeometric functions.

Although isogeometric analysis is based on NURBS, we will restrict ourselves to rational tensor-product Bézier patches for all the examples presented in this paper, especially in Subsections 3.2 and 4.2. This is not a restriction since any NURBS patch (i.e. piecewise rational) can be represented as a collection of rational patches (see e.g. Borden et al., 2011). Hence we consider $\mathcal{S}^{\mathbf{p}} = \Pi^{\mathbf{p}}$, where $\Pi^{\mathbf{p}}$ is the space of polynomials of degree $\mathbf{p}$. The Bernstein polynomials

$$B_{\mathbf{i}}(\mathbf{u}) = \prod_{j=1}^{n} \binom{p_j}{i_j} (1-u_j)^{p_j - i_j} u_j^{\,i_j}$$

for $\mathbf{u} = (u_1, \ldots, u_n)$ and

$$\mathbf{i} \in \mathbb{I} = \{(i_1, \ldots, i_n) : 0 \leq i_j \leq p_j \ \forall\ 1 \leq j \leq n\}$$

form a basis of the space $\Pi^{\mathbf{p}}$.

3. Derivatives of isogeometric functions for $n = d$

In this section we assume that the geometry mapping takes the form $\mathbf{r} : \mathbf{B} \subset \mathbb{R}^d \to \Omega \subset \mathbb{R}^d$. We derive a representation of the partial derivatives of an isogeometric function $f \in \mathcal{V}^{\mathbf{p}}$ with respect to the coordinates in physical space x_i for $i \in \{1, \ldots, d\}$. More precisely, we define an operator D_i, such that $D_i\tilde{\mathbf{r}}$ represents the graph of $\partial_i f$, similar to the representation in (4).

3.1. Derivative formula for $n = d$

Note that the graph $\mathbf{f}$ of an isogeometric function f is a hyper-surface in $\mathbb{R}^{d+1}$. In the following lemma we recall the homogeneous representation of the tangent space to a parametrized hyper-surface $\mathbf{f}$ at a point $\mathbf{f}(\mathbf{u})$.

Lemma 1. *Let* $\mathbf{f}$ *be a d-dimensional patch in* $\mathbb{R}^{d+1}$ *and* $\tilde{\mathbf{f}}$ *be its representation in homogeneous coordinates,*

$$\tilde{\mathbf{f}}(\mathbf{u}) = (\tilde{r}_0(\mathbf{u}), \tilde{r}_1(\mathbf{u}), \ldots, \tilde{r}_{d+1}(\mathbf{u}))^T .$$

Then

$$T_{\mathbf{u}} = \left\{\tilde{\mathbf{x}} \in \mathbb{R}^{d+2} : \ \det(\tilde{\mathbf{x}}, \tilde{\mathbf{f}}(\mathbf{u}), \partial_1\tilde{\mathbf{f}}(\mathbf{u}), \ldots, \partial_d\tilde{\mathbf{f}}(\mathbf{u})) = 0\right\}$$

is the homogeneous representation of the hyperplane tangent to the manifold at the point $\mathbf{f}(\mathbf{u})$*. We can denote the equation of the hyperplane in short by*

$$\tilde{\mathbf{g}}(\mathbf{u})^T\tilde{\mathbf{x}} = 0,$$

where $\tilde{\mathbf{g}}(\mathbf{u})$ *is a vector with entries*

$$\tilde{g}_j(\mathbf{u}) = \det(\tilde{\mathbf{e}}_j, \tilde{\mathbf{f}}(\mathbf{u}), \partial_1\tilde{\mathbf{f}}(\mathbf{u}), \ldots, \partial_d\tilde{\mathbf{f}}(\mathbf{u})), \tag{5}$$

for $0 \leq j \leq d + 1$*. Here* $\tilde{\mathbf{e}}_j$ *is the j-th homogeneous unit vector,* $\tilde{\mathbf{e}}_0 = (1, 0, \ldots, 0)^T$*,* $\tilde{\mathbf{e}}_1 = (0, 1, 0, \ldots, 0)^T$*, etc. Note that*

$$\partial_i\tilde{\mathbf{f}}(\mathbf{u}) = \frac{\partial\tilde{\mathbf{f}}}{\partial u_i}(\mathbf{u})$$

is the partial derivative with respect to the i-th parameter u_i*.*

We use this well-known result from projective geometry to derive a representation of the derivatives of isogeometric functions as rational hyper-surface patches.

Theorem 2. *Given a function* $f \in \mathcal{V}$ *whose graph is represented by*

$$\tilde{\mathbf{f}}(\mathbf{u}) = (\tilde{r}_0(\mathbf{u}), \tilde{r}_1(\mathbf{u}), \ldots, \tilde{r}_d(\mathbf{u}), \tilde{r}_{d+1}(\mathbf{u}))^T .$$

Then the graph of the partial derivative $\partial_i f = \frac{\partial f}{\partial x_i}$ *is represented by the parametric hyper-surface*

$$D_i\tilde{\mathbf{f}} = (\tilde{r}_0\tilde{g}_{d+1}, \tilde{r}_1\tilde{g}_{d+1}, \ldots, \tilde{r}_d\tilde{g}_{d+1}, -\tilde{r}_0\tilde{g}_i)^T ,$$

where

$$\tilde{g}_j = \det(\tilde{\mathbf{e}}_j, \tilde{\mathbf{f}}, \partial_1\tilde{\mathbf{f}}, \ldots, \partial_d\tilde{\mathbf{f}}),$$

as in (5) in Lemma 1.

Proof. Fix the parameter value to be $\mathbf{u}$. Lemma 1 states that the tangent hyperplane of the hyper-surface $\tilde{\mathbf{f}}$ at $\mathbf{u}$ is given by

$$T_{\mathbf{u}} = \left\{\tilde{\mathbf{x}} \in \mathbb{R}^{d+2} : \ \det(\tilde{\mathbf{x}}, \tilde{\mathbf{f}}(\mathbf{u}), \partial_1\tilde{\mathbf{f}}(\mathbf{u}), \ldots, \partial_d\tilde{\mathbf{f}}(\mathbf{u})) = 0\right\}.$$

Let $T_1 f(\mathbf{x})$ be the first order Taylor approximation of f at $\mathbf{r}(\mathbf{u})$. Inserting $(1, x_1, \ldots, x_d, T_1 f(\mathbf{x}))^T$ into the equation of the tangent plane leads to

$$\tilde{g}_0(\mathbf{u}) + \sum_{j=1}^{d} \tilde{g}_j(\mathbf{u})x_j + \tilde{g}_{d+1}(\mathbf{u})T_1 f(\mathbf{x}) = 0,$$

with $\tilde{g}_j$ as in (5). Resolving for $T_1 f(\mathbf{x})$ leads to

$$T_1 f(\mathbf{x}) = -\frac{\tilde{g}_0(\mathbf{u})}{\tilde{g}_{d+1}(\mathbf{u})} - \sum_{j=1}^{d} \frac{\tilde{g}_j(\mathbf{u})}{\tilde{g}_{d+1}(\mathbf{u})} x_j$$

hence

$$\frac{\partial T_1 f}{\partial x_i}(\mathbf{x}) = -\frac{\tilde{g}_i(\mathbf{u})}{\tilde{g}_{d+1}(\mathbf{u})}.$$

Since $T_1 f(\mathbf{x})$ is the first order Taylor approximation of f at the point $\mathbf{x} = \mathbf{r}(\mathbf{u})$,

$$\frac{\partial f}{\partial x_i}(\mathbf{r}(\mathbf{u})) = \frac{\partial T_1 f}{\partial x_i}(\mathbf{x}) = -\frac{\tilde{g}_i(\mathbf{u})}{\tilde{g}_{d+1}(\mathbf{u})}.$$

Therefore the graph of the derivative in Cartesian coordinates fulfills

$$\left(\frac{\tilde{r}_1(\mathbf{u})}{\tilde{r}_0(\mathbf{u})}, \ldots, \frac{\tilde{r}_d(\mathbf{u})}{\tilde{r}_0(\mathbf{u})}, \frac{\partial f}{\partial x_i}(\mathbf{r}(\mathbf{u}))\right)^T = \left(\frac{\tilde{r}_1(\mathbf{u})}{\tilde{r}_0(\mathbf{u})}, \ldots, \frac{\tilde{r}_d(\mathbf{u})}{\tilde{r}_0(\mathbf{u})}, -\frac{\tilde{g}_i(\mathbf{u})}{\tilde{g}_{d+1}(\mathbf{u})}\right)^T$$

which directly leads to the desired representation in homogeneous coordinates. □

Considering the graph of an isogeometric function f as a d-dimensional patch in $\mathbb{R}^{d+1}$, this theorem gives a direct representation of the graph of a partial derivative of f, again as a d-dimensional patch in $\mathbb{R}^{d+1}$.

Let $\mathcal{S}^{\mathbf{p},\mathbf{s}}$ be a tensor-product spline space of degree $\mathbf{p} = (p_1, \ldots, p_d)$ and smoothness $\mathbf{s} = (s_1, \ldots, s_d)$ in $\mathbf{u}$. The smoothness $\mathbf{s}$ denotes the continuity along consecutive elements. Given an isogeometric function $f \in \mathcal{V}^{\mathbf{p},\mathbf{s}}(\Omega)$ corresponding to the spline space $\mathcal{S}^{\mathbf{p},\mathbf{s}}$. Assume that $\mathbf{r}$ is a regular mapping, then the derivative $\partial_i f$ is again an isogeometric function, which fulfills $\partial_i f \in \mathcal{V}^{\hat{\mathbf{p}},\hat{\mathbf{s}}}(\Omega)$, with $\hat{\mathbf{p}} = (d+2)\mathbf{p} - \mathbf{1}$ and $\hat{\mathbf{s}} = \mathbf{s} - \mathbf{1}$. The space $\mathcal{V}^{\hat{\mathbf{p}},\hat{\mathbf{s}}}(\Omega)$ is the space of isogeometric functions corresponding to $\mathcal{S}^{\hat{\mathbf{p}},\hat{\mathbf{s}}}$. Since $\mathcal{S}^{\hat{\mathbf{p}},\hat{\mathbf{s}}} \subset \mathcal{P}(\mathbf{B})$, we can directly conclude that the space $\mathcal{V}(\Omega)$ of all isogeometric functions on Ω is closed under differentiation.

The bounds on the degree and smoothness are derived from the representation of Theorem 2 and are stated for generic functions on generic geometries. For certain choices of the function f or of the geometry mapping $\mathbf{r}$, the degree of the derivatives may be decreased and/or the smoothness may be increased. In the following we use the representation of the partial derivatives from Theorem 2 to analyze the smoothness properties of isogeometric functions defined on singular patches. Note that the smoothness may be decreased when singularities are present.

3.2. Isogeometric functions on singular patches for $n = d$

In this section we use the results from Theorem 2 to analyze the order of smoothness of isogeometric functions defined on patches containing singularities. In the following we assume to have given a homogeneous representation of a rational patch in Bernstein-Bézier form. Since any NURBS patch can be split into a collection of rational patches, we restrict ourselves to rational patches. The set $\mathrm{D}_0 \subset \bar{\mathbf{B}}$ denotes the set of singular points of the geometry mapping $\mathbf{r}$. Here $\bar{\mathbf{B}}$ is the closure of $\mathbf{B}$. Let $\det \nabla_{\mathbf{u}} \mathbf{r}(\mathbf{u}) \geq 0$ for all $\mathbf{u} \in \bar{\mathbf{B}}$ and $\det \nabla_{\mathbf{u}} \mathbf{r}(\mathbf{u}) = 0$ if and only if $\mathbf{u} \in \mathrm{D}_0$. Note that the Jacobian determinant $\det \nabla_{\mathbf{u}} \mathbf{r}(\mathbf{u})$ is a rational function of the form

$$\det \nabla_{\mathbf{u}} \mathbf{r}(\mathbf{u}) = (-1)^{d+1} \frac{\tilde{g}_{d+1}(\mathbf{u})}{(\tilde{r}_0(\mathbf{u}))^{d+1}},$$

where the numerator $\tilde{g}_{d+1}(\mathbf{u})$ is the function defined in (5). We assume that $\tilde{r}_0(\mathbf{u}) > 0$ for all $\mathbf{u} \in \bar{\mathbf{B}}$. Consequently, the set D_0 is the zero set of the multivariate polynomial $\tilde{g}_{d+1}(\mathbf{u})$. In the following we present two types of model cases of singular planar patches. We will derive $\mathscr{C}^1(\bar{\Omega})$ and $\mathscr{C}^2(\bar{\Omega})$ smoothness conditions for isogeometric functions defined on these model patches. Note that we consider the non-standard notion of $\mathscr{C}^k(\bar{\Omega})$ smoothness on a closed domain $\bar{\Omega}$.

Definition 3. Let $\Omega \subset \mathbb{R}^d$ be an open set. We say that $f \in \mathscr{C}^k(\bar{\Omega})$ if and only if $f \in C^k(\Omega)$ and the derivatives of f up to order k can be extended continuously to the boundary $\partial\Omega$ of the domain Ω. We call such a function f a $\mathscr{C}^k$ *smooth* function. Here the space $C^k(\Omega)$ denotes the standard definition of C^k continuity on an open domain Ω.

In Model Case A we deal with singularities caused by collapsing control points. In Model Case B we are considering singularities caused by parallel tangents at a corner of the parameter domain.

In both cases we compare bilinear parametrizations with parametrizations that are perturbed in the interior.

Model Case A: Collapsing edge

In the following we are studying a model patch, where one edge of the parameter domain is projected onto one point in the physical domain. As a model case we consider a bi-quadratic patch, which is a degree-elevated bilinear patch. Moreover, we consider a perturbation of this patch and compare the smoothness properties of the isogeometric functions defined on these patches.

Given an isogeometric function $f \in \mathcal{V}^{(2,2)}$ represented by its graph surface

$$\tilde{\mathbf{f}}(\mathbf{u}) = \sum_{i_1=0}^{2} \sum_{i_2=0}^{2} \tilde{\mathbf{f}}_{(i_1,i_2)} B_{i_1,i_2}(u_1, u_2)$$

with control points

$\tilde{\mathbf{f}}_{(i_1,i_2)}$	$i_2 = 0$	$i_2 = 1$	$i_2 = 2$
$i_1 = 0$	$(1, 0, 0, f_{(0,0)})^T$	$(1, 0, 0, f_{(0,1)})^T$	$(1, 0, 0, f_{(0,2)})^T$
$i_1 = 1$	$(1, 2, 0, f_{(1,0)})^T$	$(1, 1, 1, f_{(1,1)})^T$	$(1, 0, 2, f_{(1,2)})^T$
$i_1 = 2$	$(1, 4, 0, f_{(2,0)})^T$	$(1, 2, 2, f_{(2,1)})^T$	$(1, 0, 4, f_{(2,2)})^T$

and tensor-product Bernstein polynomials $B_{i_1,i_2}(u_1, u_2)$ of degree $(2, 2)$, i.e.

$$B_{i_1,i_2}(u_1, u_2) = \binom{2}{i_1}\binom{2}{i_2}(1 - u_1)^{2-i_1} u_1^{i_1} (1 - u_2)^{2-i_2} u_2^{i_2}.$$

We call such a patch $\tilde{\mathbf{f}}$ a patch with a *collapsing edge*. Moreover, we consider a function f^δ on a perturbed patch $\tilde{\mathbf{f}}^\delta$ with

$$\begin{aligned} \tilde{\mathbf{f}}^\delta_{(1,1)} &= \tilde{\mathbf{f}}_{(1,1)} + (0, \delta_1, \delta_2, 0)^T && \text{and} \\ \tilde{\mathbf{f}}^\delta_{\mathbf{i}} &= \tilde{\mathbf{f}}_{\mathbf{i}} && \text{otherwise.} \end{aligned}$$

Figure 3 contains two examples of a patch with a collapsing edge. The triangle on the left depicts the control point grid for the unperturbed patch, the triangle on the right depicts one possibility of a perturbed patch.

Using the representation from Theorem 2, we conclude that $f \in \mathscr{C}^1(\bar{\Omega})$ if and only if

$$\begin{aligned} f_{(0,0)} = f_{(0,1)} &= f_{(0,2)} \\ 2f_{(1,1)} - f_{(1,2)} - f_{(1,0)} &= 0. \end{aligned} \tag{6}$$

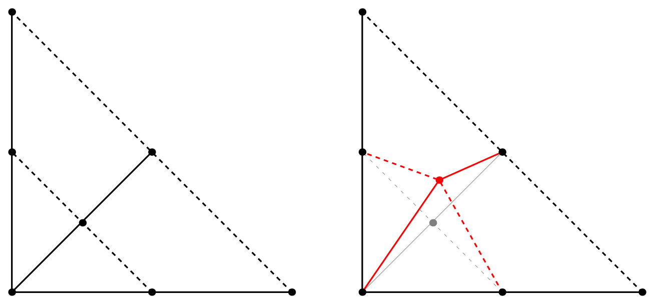

Figure 3: Model case A - geometry mapping of an unperturbed (left) and perturbed patch (right)

This relation can be proven be analyzing the derivative patches $D_i\tilde{\mathbf{f}}$ for $i = 1, \ldots, d$. In order to get a $\mathscr{C}^1$ smooth function, all the surfaces $D_i\tilde{\mathbf{f}}$ must be continuous, when projected back to Cartesian coordinates. When considering $\mathscr{C}^2$ smoothness, we get the same result, i.e. $f \in \mathscr{C}^2(\bar{\Omega})$ if and only if equation (6) is fulfilled. Note that, when dealing with singular patches, the control point representation of the derivative $D_i\tilde{\mathbf{f}}$ might contain points at infinity or basepoints. A homogeneous coordinate vector $\tilde{\mathbf{f}}$ corresponds to a point at infinity, iff $\tilde{f}_0 = 0$ and $\left\|\tilde{\mathbf{f}}\right\| \neq 0$. A basepoint is a point with $\tilde{\mathbf{f}} = (0, \ldots, 0)^T$. Hence, if $\mathbf{r}$ contains singularities, the derivative patch does not necessarily fulfill $D_i\tilde{\mathbf{f}} : \mathbf{B} \to \mathbb{R}\backslash\{0\} \times \mathbb{R}^{d+1}$ but may be extended to $D_i\tilde{\mathbf{f}} : \mathbf{B} \to \mathbb{R}^{d+2}$. If one of the partial derivatives of f contains a point at infinity in its graph representation, the function f is not in $\mathscr{C}^1$. However, if basepoints occur, this is not necessarily the case. For smooth patches containing basepoints we refer to Krasauskas (2002) on toric surface patches. The singular patch presented here is also related to the work by Reif (1995) on TURBS.

Let us now analyze the perturbed patch $\tilde{\mathbf{f}}^\delta$. Considering $\mathscr{C}^1$ smoothness, the result is similar. We conclude that $f^\delta \in \mathscr{C}^1(\bar{\Omega})$ if and only if

$$f_{(0,0)} = f_{(0,1)} = f_{(0,2)} \tag{7}$$

$$\delta_1(f_{(0,0)} - f_{(1,0)}) + \delta_2(f_{(0,0)} - f_{(1,2)}) + 2f_{(1,1)} - f_{(1,2)} - f_{(1,0)} = 0. \tag{8}$$

The second equation is equivalent to

$$\det \begin{pmatrix} 1 & 0 & 0 & f_{(0,0)} \\ 1 & 2 & 0 & f_{(1,0)} \\ 1 & 1+\delta_1 & 1+\delta_2 & f_{(1,1)} \\ 1 & 0 & 2 & f_{(1,2)} \end{pmatrix} = 0.$$

This is exactly the same condition as in the non-perturbed case ($\delta = (0, 0)$). The geometric interpretation of this condition is, that the Cartesian control points $\mathbf{f}_{(0,0)}$, $\mathbf{f}_{(0,1)}$, $\mathbf{f}_{(0,2)}$, $\mathbf{f}_{(1,0)}$, $\mathbf{f}_{(1,1)}$ and $\mathbf{f}_{(1,2)}$

must be coplanar. However, when analyzing higher order smoothness, the situation is different. From Theorem 2 we conclude that $f^\delta \in \mathscr{C}^2(\bar{\Omega})$ if and only if equations (7), (8) and

$$\begin{aligned} \delta_1(f_{(1,0)} - f_{(2,0)}) + \delta_2(f_{(1,0)} - f_{(2,1)}) + 2f_{(1,1)} - f_{(1,2)} - f_{(1,0)} &= 0 \quad (9) \\ \delta_1(f_{(0,0)} - f_{(1,0)} + f_{(1,2)} - f_{(2,1)}) + \delta_2(f_{(0,0)} - f_{(2,2)}) + 4f_{(1,1)} - 2f_{(1,2)} - 2f_{(1,0)} &= 0 \quad (10) \end{aligned}$$

are valid. One can observe that equations (8), (9) and (10) are equivalent if $(\delta_1, \delta_2) = (0, 0)$, where the system simplifies to (6).

Model case B: Parallel tangents

In the following we study a model patch, where the images of two edges in the parameter domain have parallel tangents in one point of the physical domain. As a model case we again consider a bi-quadratic patch, as a degree-elevated bilinear patch. Similar to Model Case A we also consider a perturbed patch and compare it to the unperturbed patch.

Given an isogeometric function $f \in \mathcal{V}^{(2,2)}$ represented by its graph surface

$$\tilde{\mathbf{f}}(u_1, u_2) = \sum_{i_1=0}^{2} \sum_{i_2=0}^{2} \tilde{\mathbf{f}}_{(i_1,i_2)} B_{i_1,i_2}(u_1, u_2)$$

with control points

$\tilde{\mathbf{f}}_{(i_1,i_2)}$	$i_2 = 0$	$i_2 = 1$	$i_2 = 2$
$i_1 = 0$	$(1, 2, 2, f_{(0,0)})^T$	$(1, 3, 1, f_{(0,1)})^T$	$(1, 4, 0, f_{(0,2)})^T$
$i_1 = 1$	$(1, 1, 3, f_{(1,0)})^T$	$(1, \frac{3}{2}, \frac{3}{2}, f_{(1,1)})^T$	$(1, 2, 0, f_{(1,2)})^T$
$i_1 = 2$	$(1, 0, 4, f_{(2,0)})^T$	$(1, 0, 2, f_{(2,1)})^T$	$(1, 0, 0, f_{(2,2)})^T$

and tensor-product Bernstein polynomials $B_{i_1,i_2}(u_1, u_2)$ of degree $(2, 2)$. In this case the patch $\tilde{\mathbf{f}}$ is called a patch with *parallel tangents*. As in Model Case A we again consider a function f^δ on a perturbed patch $\tilde{\mathbf{f}}^\delta$ with

$$\begin{aligned} \tilde{\mathbf{f}}^\delta_{(1,1)} &= \tilde{\mathbf{f}}_{(1,1)} + (0, \delta_1, \delta_2, 0)^T && \text{and} \\ \tilde{\mathbf{f}}^\delta_{\mathbf{i}} &= \tilde{\mathbf{f}}_{\mathbf{i}} && \text{otherwise.} \end{aligned}$$

Similar to the previous case, Figure 4 depicts an unperturbed patch with parallel tangents on the left and a perturbed patch on the right.

Similar to the analysis of patches with an collapsing edge, we use the representation from Theorem 2 to conclude that $f \in \mathscr{C}^1(\bar{\Omega})$ if and only if

$$\begin{aligned} 2f_{(0,0)} &= f_{(0,1)} + f_{(1,0)} \\ f_{(2,0)} - f_{(0,2)} &= 2f_{(1,0)} - 2f_{(0,1)}. \end{aligned} \quad (11)$$

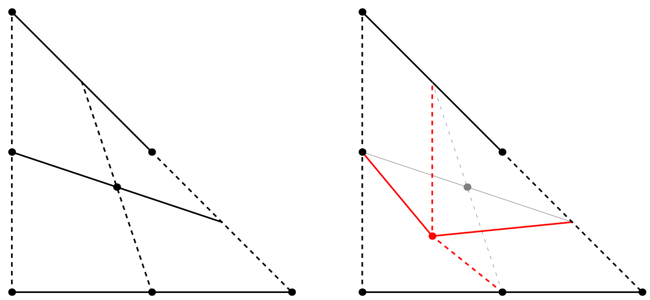

Figure 4: Model case B - geometry mapping of an unperturbed (left) and perturbed patch (right)

Moreover, $f \in \mathscr{C}^2(\bar{\Omega})$ if and only if equations (11) and

$$4f_{(0,1)} - 2f_{(0,2)} = 4f_{(1,1)} - f_{(2,1)} - f_{(1,2)} \tag{12}$$

are fulfilled. When considering the perturbed patch, the $\mathscr{C}^1$ smoothness condition is the same as in the unperturbed case, i.e. $f^\delta \in \mathscr{C}^1(\bar{\Omega})$ if and only if (11) is fulfilled. However, when considering $\mathscr{C}^2$ we get that $f^\delta \in \mathscr{C}^2(\bar{\Omega})$ if and only if (11) and

$$4\delta_2 f_{(0,0)} + (4 + 2\delta_1 - 2\delta_2)f_{(0,1)} - 2f_{(0,2)} = 4f_{(1,1)} - (1 - \delta_1 - \delta_2)f_{(2,1)} - (1 - \delta_1 - \delta_2)f_{(1,2)} \tag{13}$$

$$(f_{(0,2)} + f_{(2,0)})(\delta_1 - \delta_2) = (4\delta_1 - 6\delta_2)f_{(0,1)} + 2\delta_1 f_{(1,0)} + (\delta_1 + \delta_2)(f_{(1,2)} - f_{(2,1)}) \tag{14}$$

are satisfied. Again, if $(\delta_1, \delta_2) = (0, 0)$ then equation (13) simplifies to equation (12) and equation (14) is satisfied since both sides vanish.

As we have seen in both examples, the additional conditions restricting the smoothness only vanish for a specific positioning of the inner control point. Hence, the smoothness properties are very sensitive with respect to the exact choice of the parametrization. In any application where the parametrization is given as some free-form-object, the possible additional smoothness is very hard to attain.

4. Isogeometric functions on patches with $n < d$

In this section we are extending the result from Section 3 to functions on n-manifolds embedded into $\mathbb{R}^d$, where $n < d$. We derive a representation of the intrinsic gradient of an isogeometric function $f \in \mathcal{V}$ in Subsection 4.1. In Subsection 4.2 we focus on the case $n = 2$ and $d = 3$. We apply the derivative formula to a function which is defined on a part of a spherical cap and derive smoothness conditions in that case.

4.1. Intrinsic gradient of an isogeometric function for the case $n < d$

We consider an n-dimensional patch $\mathbf{r}(\mathbf{u})$ with homogeneous representation

$$\tilde{\mathbf{r}}(\mathbf{u}) = (\tilde{r}_0(\mathbf{u}), \tilde{r}_1(\mathbf{u}), \ldots, \tilde{r}_d(\mathbf{u}))^T$$

embedded into $\mathbb{R}^d$, with $n < d$, and an isogeometric function f defined on $\mathbf{r}$. Let $\mathbf{x}_0 \in \mathbb{R}^d$ be a point on the patch $\mathbf{r}$. Let F be an extension of f to an open neighborhood $\mathcal{N}_{\mathbf{x}_0} \subset \mathbb{R}^d$ of $\mathbf{x}_0$. The intrinsic gradient $\nabla_{\mathbf{r}} f$ of f is given by

$$\nabla_{\mathbf{r}} f(\mathbf{x}_0) = \Pi_{\mathbf{r}} \left(\nabla_{\mathbf{x}} F(\mathbf{x}_0) \right),$$

where $\nabla_{\mathbf{x}} F$ is the gradient of F with respect to $\mathbf{x}$ in $\mathcal{N}_{\mathbf{x}_0}$. The operator $\Pi_{\mathbf{r}}$ yields the projection of a vector onto the tangent space of the patch $\mathbf{r}$ in $\mathbf{x}_0$. When considering a surface in space (i.e. $n = 2$, $d = 3$), this projection can be described by

$$\Pi_{\mathbf{r}} \left(\nabla_{\mathbf{x}} F \right) = \nabla_{\mathbf{x}} F - \left((\nabla_{\mathbf{x}} F)^T \mathbf{n} \right) \mathbf{n},$$

where $\mathbf{n}$ is the unit normal vector of $\mathbf{r}$ in $\mathbf{x}_0$. One can show that the projection of the gradient onto the tangent space does not depend on the extension F of the function f, but only on the values of F on the patch (i.e. $\nabla_{\mathbf{r}} f$ only depends on f). Therefore the intrinsic gradient is well-defined. Here we are following the notion from Šír et al. (2008). For further details on the presented objects from differential geometry we refer to Kreyszig (1991).

We now construct an extension F of f such that the intrinsic gradient is simple to compute. In a first step the n-dimensional patch $\tilde{\mathbf{r}}(\mathbf{u})$, with $\mathbf{u} \in \mathbb{R}^n$, is extended to a d-dimensional patch in $\mathbb{R}^d$. We assume to have given a basis $\mathbf{n}^{n+1}(\mathbf{u}), \ldots, \mathbf{n}^d(\mathbf{u})$ of the normal space at $\mathbf{u}$, i.e. $(\partial_k \mathbf{r})^T \mathbf{n}^j = 0$ for all $k = 1, \ldots, n$ and $j = n+1, \ldots, d$, where $\mathbf{r}$ is the Cartesian representation of $\tilde{\mathbf{r}}$. Moreover let $\mathbf{N} = (\mathbf{n}^{n+1}, \ldots, \mathbf{n}^d)$. We denote by

$$\mathbf{F}(\mathbf{u}, \mathbf{u}^+) = \begin{pmatrix} \mathbf{r}(\mathbf{u}) \\ r_{d+1}(\mathbf{u}) \end{pmatrix} + \begin{pmatrix} \mathbf{u}^+ \mathbf{N}(\mathbf{u}) \\ 0 \end{pmatrix}$$

the extended graph of f, where $\mathbf{u}^+ = (u_{n+1}, \ldots, u_d)$. Here $\mathbf{u}^+ \mathbf{N}(\mathbf{u})$ denotes the sum $\sum_{j=n+1}^{d} u_j \mathbf{n}^j(\mathbf{u})$.

It is not always clear how to choose a suitable basis for the normal space. Usually the goal is to have a representation of low degree. For the case $d = 3$ and $n = 2$ the normal space may be given by the standard (Cartesian) normal vector $\partial_1 \mathbf{r} \times \partial_2 \mathbf{r}$. However, in general the choice of normal vectors is not unique. In the following we sketch an approach to generate the homogeneous representation of the normal space directly.

Transforming the extension F to homogeneous coordinates leads to

$$\tilde{\mathbf{F}}(\mathbf{u},\mathbf{u}^+) = \begin{pmatrix} \tilde{\mathbf{r}}(\mathbf{u}) \\ \tilde{r}_{d+1}(\mathbf{u}) \end{pmatrix} + \mathbf{u}^+\tilde{\mathbf{N}}(\mathbf{u}), \tag{15}$$

where $\tilde{\mathbf{N}} = (\tilde{\mathbf{n}}^{n+1}, \ldots, \tilde{\mathbf{n}}^d)$.

One way to get a set $\tilde{\mathbf{N}}$ of normal vectors $\tilde{\mathbf{n}}^k$, $n+1 \leq k \leq d$, of low degree is to compute the exterior product (generalized cross product)

$$\tilde{\mathbf{r}} \wedge \partial_1\tilde{\mathbf{r}} \wedge \ldots \wedge \partial_n\tilde{\mathbf{r}} \wedge \tilde{\mathbf{q}}_{n+1} \wedge \ldots \wedge \tilde{\mathbf{q}}_{d-1},$$

of $\tilde{\mathbf{r}}$ and the partial derivatives of $\tilde{\mathbf{r}}$ with arbitrary points $\tilde{\mathbf{q}}_{n+1}, \ldots, \tilde{\mathbf{q}}_{d-1}$. We refer to Frankel (1997) for a detailed description of this operation. This product describes a hyperplane that contains the tangential space of the patch $\tilde{\mathbf{r}}$. Hence the normal vector to this hyperplane is a possible choice for the vectors $\tilde{\mathbf{n}}^k$, with $n+1 \leq k \leq d$. Applying this procedure $(d-n)$ times leads to a set of $(d-n)$ normal vectors of degree $\mathbf{p_n} \leq (n+1)\mathbf{p} - \mathbf{1}$.

From the equation of the extension (15) it follows that, $\tilde{\mathbf{F}}(\mathbf{u},\mathbf{0}) = (\tilde{r}_0(\mathbf{u}), \ldots, \tilde{r}_{d+1}(\mathbf{u}))^T = \tilde{\mathbf{f}}(\mathbf{u})$ represents the graph of the isogeometric function defined on the patch $\tilde{\mathbf{r}}$.

Theorem 4. *Given a function $f \in \mathcal{V}^{\mathbf{p}}$ whose extended graph is represented by $\tilde{\mathbf{F}}(\mathbf{u},\mathbf{u}^+)$ as in (15), where $(\mathbf{u},\mathbf{u}^+) \in \mathbf{B} \subset \mathbb{R}^d$. Let $\nabla_{\mathbf{r}} f$ be the intrinsic gradient of f. Then the graph of the i-th component $\nabla_{\mathbf{r},i} f$ of the intrinsic gradient is represented by the parametric surface*

$$D_i\tilde{\mathbf{f}} = (\tilde{r}_0\tilde{g}_{d+1}, \tilde{r}_1\tilde{g}_{d+1}, \ldots, \tilde{r}_d\tilde{g}_{d+1}, -\tilde{r}_0\tilde{g}_i)^T,$$

where

$$\tilde{g}_j = \det(\tilde{\mathbf{e}}_j, \tilde{\mathbf{f}}, \partial_1\tilde{\mathbf{f}}, \ldots, \partial_n\tilde{\mathbf{f}}, \tilde{\mathbf{n}}^{n+1}, \ldots, \tilde{\mathbf{n}}^d).$$

Proof. Applying Theorem 2 to the extended graph $\tilde{\mathbf{F}}$, we get

$$D_i\tilde{\mathbf{F}} = \left(\tilde{F}_0\tilde{q}_{d+1}, \tilde{F}_1\tilde{q}_{d+1}, \ldots, \tilde{F}_d\tilde{q}_{d+1}, -\tilde{F}_0\tilde{q}_i\right)^T,$$

with

$$\tilde{q}_j = \det(\tilde{\mathbf{e}}_j, \tilde{\mathbf{F}}, \partial_1\tilde{\mathbf{F}}, \ldots, \partial_d\tilde{\mathbf{F}}).$$

The hyper-surface $D_i\tilde{\mathbf{F}}$ represents the i-th component of the gradient $\nabla_{\mathbf{x}}F$. Restricting the gradient $\nabla_{\mathbf{x}}F$ to the patch $\mathbf{r}$ is equivalent to restricting the parameter value to $(\mathbf{u},\mathbf{u}^+) = (\mathbf{u},\mathbf{0})$. Hence,

$$D_i\tilde{\mathbf{F}}\,|_{\{\mathbf{u}^+=\mathbf{0}\}} = (\tilde{r}_0\tilde{g}_{d+1}, \tilde{r}_1\tilde{g}_{d+1}, \ldots, \tilde{r}_d\tilde{g}_{d+1}, -\tilde{r}_0\tilde{g}_i)^T,$$

with

$$\tilde{g}_j = \tilde{q}_j\,|_{\{\mathbf{u}^+=\mathbf{0}\}} = \det(\tilde{\mathbf{e}}_j, \tilde{\mathbf{f}}, \partial_1\tilde{\mathbf{f}}, \ldots, \partial_n\tilde{\mathbf{f}}, \tilde{\mathbf{n}}^{n+1}, \ldots, \tilde{\mathbf{n}}^d).$$

We can split the gradient $\nabla_{\mathbf{x}} F$ into two parts, i.e.

$$\nabla_{\mathbf{x}} F = \Pi_{\mathbf{r}} (\nabla_{\mathbf{x}} F) + \Pi_{\mathbf{n}} (\nabla_{\mathbf{x}} F),$$

where $\Pi_{\mathbf{r}}$ is the projection onto the tangent space and $\Pi_{\mathbf{n}}$ is the projection onto the normal space. However, due to the construction of $\mathbf{F}$, the normal component $\Pi_{\mathbf{n}} (\nabla_{\mathbf{x}} F)$ vanishes everywhere. Hence,

$$\nabla_{\mathbf{x}} F = \Pi_{\mathbf{r}} (\nabla_{\mathbf{x}} F) = \nabla_{\mathbf{r}} f$$

which concludes the proof. □

Assuming we can represent the homogeneous normal vectors $\tilde{\mathbf{n}}^j$ as polynomials of degree $\mathbf{p_n}$, the degree of the derivative patch is bounded by $(d - n)\mathbf{p_n} + (n + 2)\mathbf{p} - \mathbf{1}$.

Higher order derivatives of f can be computed recursively. The Hessian of f, as a tensor in $\mathbb{R}^{d\times d}$ can be computed by applying the gradient operator $\nabla_{\mathbf{r}}$ again to each component of $\nabla_{\mathbf{r}} f$, i.e. if $\nabla_{\mathbf{r},i} f$ is the i-th component of the intrinsic gradient, then

$$\mathrm{Hess}(f) = \begin{pmatrix} | & | & | \\ \nabla_{\mathbf{r}}(\nabla_{\mathbf{r},1} f) & \nabla_{\mathbf{r}}(\nabla_{\mathbf{r},2} f) & \nabla_{\mathbf{r}}(\nabla_{\mathbf{r},3} f) \\ | & | & | \end{pmatrix} \tag{16}$$

is the Hessian of f. We refer to Šír et al. (2008) for further details.

In the following subsections we consider the case of a surface in $\mathbb{R}^3$ ($d = 3$ and $n = 2$), more precisely we consider a spherical cap.

4.2. Singularly parametrized surface in $\mathbb{R}^3$

When considering a surface in $\mathbb{R}^3$, the normal space in $\mathbf{u}$ is spanned by the normal vector $\mathbf{n}(\mathbf{u}) = \partial_1 \mathbf{r}(\mathbf{u}) \times \partial_2 \mathbf{r}(\mathbf{u})$. In homogeneous coordinates this leads to

$$\tilde{\mathbf{n}} = -\begin{pmatrix} 0 \\ \det(\tilde{\mathbf{e}}_1, \tilde{\mathbf{r}}, \partial_1 \tilde{\mathbf{r}}, \partial_2 \tilde{\mathbf{r}}) \\ \det(\tilde{\mathbf{e}}_2, \tilde{\mathbf{r}}, \partial_1 \tilde{\mathbf{r}}, \partial_2 \tilde{\mathbf{r}}) \\ \det(\tilde{\mathbf{e}}_3, \tilde{\mathbf{r}}, \partial_1 \tilde{\mathbf{r}}, \partial_2 \tilde{\mathbf{r}}) \end{pmatrix}.$$

Hence the degree of the normal vector is bounded by $\mathbf{p_n} = 3\mathbf{p} - \mathbf{1}$, leading to $\mathbf{p_n} + 4\mathbf{p} - \mathbf{1} = 7\mathbf{p} - \mathbf{2}$ as a bound on the degree of the gradient patches. Using this representation and Theorem 4 with

$$\tilde{g}_j = \det(\tilde{\mathbf{e}}_j, \tilde{\mathbf{f}}, \partial_1 \tilde{\mathbf{f}}, \partial_2 \tilde{\mathbf{f}}, \tilde{\mathbf{n}})$$

we can conclude smoothness results for surfaces embedded in $\mathbb{R}^3$.

Model Case: Spherical cap

The final example presented in this paper is a part of a spherical cap. We consider a bi-quadratic rational patch. The patch represents a function defined on one eighth of the unit sphere, with a singularity at the pole.

Given an isogeometric function f represented by its graph surface

$$\tilde{\mathbf{f}}(u_1, u_2) = \sum_{i_1=0}^{2}\sum_{i_2=0}^{2} \tilde{\mathbf{f}}_{(i_1,i_2)} B_{i_1,i_2}(u_1, u_2)$$

with control points

$\tilde{\mathbf{f}}_{(i_1,i_2)}$	$i_2 = 0$	$i_2 = 1$	$i_2 = 2$
$i_1 = 0$	$(1, 0, 0, 1, f_{(0,0)})^T$	$\frac{1}{\sqrt{2}}(1, 0, 0, 1, f_{(0,1)})^T$	$(1, 0, 0, 1, f_{(0,2)})^T$
$i_1 = 1$	$\frac{1}{\sqrt{2}}(1, 1, 0, 1, f_{(1,0)})^T$	$\frac{1}{2}(1, 1, 1, 1, f_{(1,1)})^T$	$\frac{1}{\sqrt{2}}(1, 0, 1, 1, f_{(1,2)})^T$
$i_1 = 2$	$(1, 1, 0, 0, f_{(2,0)})^T$	$\frac{1}{\sqrt{2}}(1, 1, 1, 0, f_{(2,1)})^T$	$(1, 0, 1, 0, f_{(2,2)})^T$

and tensor-product Bernstein polynomials $B_{i_1,i_2}(u_1, u_2)$ of degree $(2, 2)$.

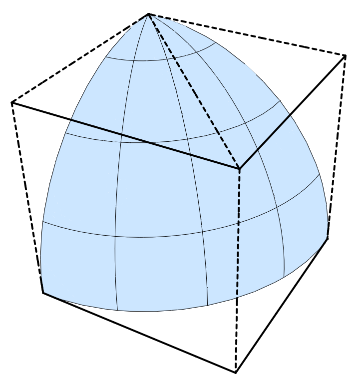

Figure 5: Spherical cap - patch and control point grid

Figure 5 depicts the spherical cap and its control point grid.

Using the representation from Theorem 2, we conclude that $f \in \mathscr{C}^1(\bar{\Omega})$ if and only if

$$\begin{aligned} f_{(0,0)} = f_{(0,1)} &= f_{(0,2)} \\ f_{(1,2)} - f_{(1,1)} + f_{(1,0)} - f_{(0,0)} &= 0. \end{aligned} \quad (17)$$

Similar to the results for Model Case A, equation (17) is equivalent to the points $\mathbf{f}_{(0,0)}$, $\mathbf{f}_{(0,1)}$, $\mathbf{f}_{(0,2)}$, $\mathbf{f}_{(1,0)}$, $\mathbf{f}_{(1,1)}$ and $\mathbf{f}_{(1,2)}$ lying in a 2-plane in $\mathbb{R}^4$. By that we mean that the corresponding Cartesian points lie on the same two-dimensional subspace of $\mathbb{R}^4$. When considering $\mathscr{C}^2$ smoothness, we get a more restrictive result.

If we apply Theorem 2 to each component of the gradient, we get $\mathscr{C}^2$ smoothness results. By $\mathscr{C}^2$ smoothness we mean, that the Hessian, as defined in (16), is smooth on the closed domain. Hence we conclude that $f \in \mathscr{C}^2(\bar{\Omega})$ if and only if equation (17) and

$$\begin{aligned} f_{(2,1)} - f_{(1,1)} + f_{(2,0)} - f_{(1,0)} &= 0 \\ f_{(2,2)} - f_{(2,0)} + f_{(1,0)} - f_{(1,2)} &= 0 \end{aligned} \tag{18}$$

are valid. In total we have given 9 variables and 5 equations, which gives 4 remaining degrees of freedom. To get a $\mathscr{C}^2$ smooth isogeometric function, one could for instance prescribe function values at $\tilde{\mathbf{r}}(\mathbf{u}_i)$ for $\mathbf{u}_1 = (0,0)$, $\mathbf{u}_2 = (1,0)$, $\mathbf{u}_3 = (1,1/2)$ and $\mathbf{u}_4 = (1,1)$.

5. Conclusion

We developed a homogeneous closed form representation of the derivatives of an isogeometric function. We took advantage of the fact that the graph of an isogeometric function defined on an n-dimensional NURBS patch in $\mathbb{R}^d$ can be interpreted as an n-dimensional patch in $\mathbb{R}^{d+1}$. The graphs of the partial derivatives of an isogeometric function are then again given as NURBS patches on the same NURBS geometry mapping. Hence the space of isogeometric functions is closed with respect to differentiation.

We applied the presented representation of the derivatives to derive smoothness conditions for isogeometric functions on patches containing singularities. The method provides a systematic approach to derive smoothness conditions of arbitrary order. The study of two model cases indicates that the smoothness conditions depend heavily on perturbations of the geometry mapping. The results developed here are strongly related to the H^1 and H^2 regularity results from Takacs and Jüttler (2011, 2012).

One possibility for future work is to extend the results to isogeometric vector fields. Consider a vector field that is defined on a surface $\mathbf{r}$ in space, i.e. $n = 2$ and $d = 3$, and assume that the vector field is tangential to the surface. Hence the vector field is orthogonal to the surface normal $\mathbf{n}_\mathbf{r}$. To represent such a vector field one may define a general vector valued isogeometric function $\mathbf{V} \in \mathbb{R}^3$ on the surface patch and enforce the orthogonality condition weakly via $\|\mathbf{V}^T \mathbf{n}_\mathbf{r}\| \approx 0$. This ansatz can then be used to compute a vector field fulfilling certain properties, such as interpolating given data. In that case a regularization of the vector field may be necessary. Using the approach proposed in this paper, a regularization functional based on the covariant derivatives of the vector field can be constructed. We refer to Kirisits et al. (2013), where a similar approach was employed to approximate the optical flow of an image on a moving surface.

Another issue related to the presented approach is the resulting degree of the derivative patches. One may be interested in reducing the degree directly or in approximating the derivative patch of high degree by a patch of lower degree.

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