An Interior Penalty Discontinuous Galerkin Finite Element Method for Quasilinear Parabolic Problems

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Abstract

In this paper, an Interior Penalty Discontinuous Galerkin finite element method (IPDG) is analyzed for approximating quasilinear parabolic equations. The equations can be characterized as perturbed parabolic p-Laplacean equations. The fully discrete scheme is obtained by applying s-stage Diagonally Implicit Runge-Kutta Methods (s-DIRK) for the time integration. The nonlinear systems of the algebraic equations appearing in s-DIRK cycles are solved by developing two low storage Picard iterative processes. A stability bound is shown for the semi-discrete IPDG solution in the broken $\|.\|_{DG,p}$-norm. Continuous in time a priori error estimates are proved in case of $p > 2$, when linear approximation space is used. A numerical test is performed in order to compare the performance of the two Picard iterative processes. Also, the results presented in the theoretical analysis are confirmed by numerical examples.

Keywords: quasilinear parabolic problems, perturbed parabolic p-Laplace problem, interior penalty discontinuous Galerkin method, stability estimates, a priori error estimates.

1. Introduction

In this paper, an Interior Penalty Discontinuous Galerkin method (IPDG) is studied for approximating solutions of quasilinear problems in L^p setting, which can be recognized as examples of the perturbed p-Laplace problem. The problems are described by nonlinear diffusion equations, where the diffusion coefficient has a standard p-exponent form, that is $(\mu+|\nabla u|)^{p-2}, \mu > 0$, with most interesting case here $\mu = 1$, [1]. Very often, these constitute the mathematical model in many practical applications, as in aerodynamic, non-Newtonian flows, plasticity and glaciology, see e.g. [2], [3].

Over the last two decades, there has been an increasing interest on devising discontinuous Galerkin (DG) methods for the numerical solution of elliptic and parabolic problems. This interest comes from the advantages of the local approximation spaces without continuity requirements that DG methods offer. Finite element methods defined on discontinuous spaces with interior penalties

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for linear elliptic problems were first analyzed in [4], [5]. These methods, for the construction of the penalty terms on the interfaces, use similar techniques as the Nitsche's treatment of introducing penalty terms for imposing Dirichlet boundary conditions. These approaches are generalized by symmetric and non-symmetric IPDG methods, see [6], [7],[8], for a comprehensive analysis of IPDG methods for linear elliptic problems. Recently, DG methods have been proposed and analyzed for applications to nonlinear elliptic problems formulated in $W^{1,2}(\Omega)$. For example, in [9], DG methods have been analyzed for second order elliptic and hyperbolic systems and in [10], DG symmetric/non-symmetric methods have been analyzed for non-Fickian diffusion problems. In [11], an incomplete IPDG is introduced for a class of second order monotone nonlinear elliptic problems and a priori error estimates are given under minimal regularity assumptions on the exact solution. We also refer [12], where a *hp*-DG method has been studied for monotone quasilinear elliptic problems. Using theory of monotone operators, the authors showed the uniqueness the DG solution and derive a priori error estimate in a mesh-dependent energy norm. Based on the already results for steady problems, IPDG methods have been proposed for solving parabolic type problems. We refer, but not limited to, the following. In [13], the first analysis of a semi-discrete IPDG method was presented for linear problems and in [14], optimal error estimates for a semi-discrete symmetric IPDG method have obtained for nonlinear parabolic problems. We also mention [15] and [16], where error estimates are discussed for fully discrete IPDG methods, and furthermore, we refer [17] where three fully discrete IPDG methods are considered and analyzed.

In contrast to the analysis of IPDG methods for elliptic problems with natural formulation in $W^{1,2}(\Omega)$, there are no contributions that are concerned with nonlinear problems formulated in $W^{1,p\neq 2}(\Omega)$, like the problem with p-exponent diffusion coefficient that is considered here. Maybe as one exception, we can refer the work presented in [18], where IPDG approximate solutions are studied for the p-Laplace equation, ($\mu = 0$). It is the purpose of this paper to make a first step in this direction.

We point out that, classical (continuous) finite element methods, for more general p-form problems, the so-called (p, δ)-structure problems, have been analyzed in the literature, see e.g. [19] and [20]. For parabolic (p, δ)-structure problems, we refer [21], where optimal convergence rates have been shown, in case of using linear finite element in space and implicit Euler scheme in time.

The IPDG scheme proposed here, see (3.8), has the same form as the IPDG scheme in [12], but here the numerical flux is appropriately re-formulated in order to be compatible with the p-nature of the problem. As a first task, stability bounds are proved in $\|.\|_{DG,p}$-norm, for the case of $\mu = 0$. Then, using the interpolation estimates presented in [20], a priori error estimates are given for the semi-discrete problem for $p > 2$, assuming conventional regularity for the exact solution. The IPDG spatial discretization, generates a nonlinear ODE system with respect to the degrees of freedom. We discretize in time this system by s-stage Diagonally Implicit Runge-Kutta methods (s-DIRK). Every cycle of the Runge-Kutta method includes the solution of nonlinear algebraic systems. Two low computational cost Picard block-iterative methods are proposed for solving the nonlinear systems, [22]. The Picard iterative methods are constructed based on the local (per element) approximation features of the IPDG method. The two different iterative methods are expected to have the same order of convergence (first order), but different performance speed, since the second one uses the latest available solution (and not the solution of the previous iteration) for updating the nonlinear parts of the system.

The outline of the paper is as follows. It begins by presenting the model problem. Then inequalities for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ are shown, which are used later to derive the continuity-monotonicity properties of the scheme. In Section 3, the IPDG method is described. Section 4 includes the formulation of the s-DIRK method for the time discretization and the description of the two Picard methods. In Section 5, a stability bound for the discrete solution of the p-Laplace problem is presented. A priori error estimates for $p > 2$ are shown in Section 6. The paper closes with the numerical tests in Section 7.

2. The model problem

Let Ω be a bounded domain in $\mathbb{R}^2$, with smooth boundary $\Gamma_D := \partial\Omega$. We consider the following scalar initial boundary value problem

$$u_t - \mathrm{div}\mathbf{A}(\nabla u) = f \qquad \textit{in } \Omega \times (0, T] \tag{2.1a}$$

$$u_0(x) = u(x, 0) \qquad \textit{in } \Omega \tag{2.1b}$$

$$u = u_D, \qquad \textit{on } \Gamma_D \times (0, T], \tag{2.1c}$$

where $(0, T]$ is the time interval, $f : \Omega \times (0, T] \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, $u_D : \Gamma_D \times (0, T] \to \mathbb{R}$ are given smooth functions. The operator $\mathbf{A}(\nabla u) : \mathbb{R}^2 \to \mathbb{R}^2$ has the form

$$\mathbf{A}(\nabla u) = (\mu + |\nabla u|)^{p-2}\nabla u, \; p > 1, \; \mu \geq 0,$$

where $|.| : \mathbb{R}^2 \to \mathbb{R}$ is the Euclidean measure and $a(\nabla u) = (\mu + |\nabla u|)^{p-2}$ is the diffusion coefficient. The nonlinear nature of the problem (2.1) comes by the appearance of $|\nabla u|$ in the diffusive coefficient and this poses numerical challenges. The IPDG methods presented so far for nonlinear elliptic equations are referred to problems where the natural formulation is given in $W^{1,2}(\Omega)$, and either $a(.)$ is uniformly bounded, e.g. [23], or $a(.)$ satisfies a monotone condition, e.g. [11], [12]. One can not applied the same methodology for the problem (2.1) which is formulated in $W^{1,p}(\Omega)$. This fact motivates the need of further analysis and of developing numerical fluxes compatible with the p-exponent form of the diffusion coefficient. The goal of this paper is to make a first step in this direction.

Assuming that $f \in C([0, T]; L^2(\Omega))$, $u_0 \in W^{1,2}(\Omega) \cap W^{1,p}(\Omega)$, we call u weak solution of (2.1), if $u \in L^\infty(0, T; W^{1,p}(\Omega)) \cap W^{1,2}(0, T; W^{1,2}(\Omega))$, $u|_{\Gamma_D} := u_D$ satisfies the following formulation for any $v \in W_0^{1,p}(\Omega)$

$$\forall t > 0, \; \int_\Omega u_t v \, dx + \int_\Omega \mathbf{A}(\nabla u) \cdot \nabla v \, dx = \int_\Omega f v \, dx, \tag{2.2}$$

where

$$L^p(0,T;V) = \{v : (0,T) \to V : \int_0^T \|v(t)\|_V^p \, dt < \infty\}.$$

The existence-uniqueness of the solution of (2.2) (even with other assumptions on the data) are ensured by means of the monotone operators theory, see e.g. [1], [24]. We refer [25], [21], for regularity assumptions on the problem data for obtaining optimal rate of convergence for finite element solutions. For the analysis here, we assume the following conventional assumptions

$$u \in W^{1,2}(0,T;W^{1,2}(\Omega)) \cap L^p(0,T;W^{s\geq 2,p}(\Omega)). \tag{2.3}$$

Through the paper $C_i, i = 1, \ldots$ will be generic constants with different values independent of crucial quantities. The explicit dependence on the problem data will be mentioned.

2.1. Helpful inequalities for vectors

Working further on the results of Chp I in [24] and [26], we prove special algebraic inequalities that are going to be used later. In the proofs, we use the function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{F}(\mathbf{a}) = (\mu + |\mathbf{a}|)^{\frac{p-2}{2}} \mathbf{a}. \tag{2.4}$$

We introduce the formula

$$\mathbf{A}(\mathbf{b}) - \mathbf{A}(\mathbf{a}) = \int_0^1 \frac{d}{dt} (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-2} (\mathbf{a} + t(\mathbf{b} - \mathbf{a}))\, dt, \tag{2.5}$$

and by an easy computation on the right hand of (2.5) we get

$$\begin{aligned}\mathbf{A}(\mathbf{b}) - \mathbf{A}(\mathbf{a}) = \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-2} (\mathbf{b} - \mathbf{a})\, dt + \\ (p-2) \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-3} \frac{1}{2} |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|^{-1} \\ 2\big(\mathbf{a} + t(\mathbf{b} - \mathbf{a}), \mathbf{b} - \mathbf{a}\big)(\mathbf{a} + t(\mathbf{b} - \mathbf{a}))\, dt.\end{aligned} \tag{2.6}$$

Multiplying (2.6) by $\mathbf{b} - \mathbf{a}$, we have

$$\begin{aligned}\big(\mathbf{A}(\mathbf{b}) - \mathbf{A}(\mathbf{a}), \mathbf{b} - \mathbf{a}\big) = |\mathbf{b} - \mathbf{a}|^2 \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-2}\, dt + \\ (p-2) \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-3} |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|^{-1} \\ \big(\mathbf{a} + t(\mathbf{b} - \mathbf{a}), \mathbf{b} - \mathbf{a}\big)^2\, dt.\end{aligned} \tag{2.7}$$

The last term on the right hand side of (2.7) is positive and for $p \geq 2$ we get

$$\big(\mathbf{A}(\mathbf{b}) - \mathbf{A}(\mathbf{a}), \mathbf{b} - \mathbf{a}\big) \geq |\mathbf{b} - \mathbf{a}|^2 \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-2}\, dt \tag{2.8a}$$

by applying Cauchy-Schwarz inequalities, we further get

$$\Big|\mathbf{A}(\mathbf{b}) - \mathbf{A}(\mathbf{a})| \geq |\mathbf{b} - \mathbf{a}| \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-2}\, dt. \tag{2.8b}$$

Also, applying Cauchy-Schwarz inequality on the last term on the right hand side of (2.6), we have

$$\begin{aligned} \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-3} |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|^{-1} |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|^2 |\mathbf{b} - \mathbf{a}| \\ \leq |\mathbf{b} - \mathbf{a}| \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-2}\, dt. \end{aligned} \tag{2.9}$$

Therefore, combining (2.9) and (2.6), we get

$$|\mathbf{A}(\mathbf{b}) - \mathbf{A}(\mathbf{a})| \leq (p-1)|\mathbf{b} - \mathbf{a}| \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-2}\, dt. \tag{2.10}$$

Recalling the forms of $\mathbf{A}$ and $\mathbf{F}$ and setting in (2.10) $p := \frac{p+2}{2}$ we obtain

$$\Big|(\mu + |\mathbf{b}|)^{\frac{p-2}{2}} \mathbf{b} - (\mu + |\mathbf{a}|)^{\frac{p-2}{2}} \mathbf{a}\Big|^2 \leq \big(\frac{p}{2}\big)^2 |\mathbf{b} - \mathbf{a}|^2 \Big(\int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{\frac{p-2}{2}}\, dt \Big)^2$$

and thus, $$|\mathbf{F}(\mathbf{b}) - \mathbf{F}(\mathbf{a})|^2 \leq \big(\frac{p}{2}\big)^2 |\mathbf{b} - \mathbf{a}|^2 \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{p-2}\, dt.$$

By (2.8a) and (2.8b), we have

$$|\mathbf{F}(\mathbf{b}) - \mathbf{F}(\mathbf{a})|^2 \leq C(p)\big(\mathbf{A}(\mathbf{b}) - \mathbf{A}(\mathbf{a}), \mathbf{b} - \mathbf{a}\big), \tag{2.12a}$$

$$|\mathbf{F}(\mathbf{b}) - \mathbf{F}(\mathbf{a})|^2 \leq |\mathbf{A}(\mathbf{b}) - \mathbf{A}(\mathbf{a})|^2. \tag{2.12b}$$

Keeping $p > 2$ and using that $(\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{2\frac{p-2}{2}} \leq 2 \max\{(\mu + |\mathbf{a}|)^{\frac{p-2}{2}}, (\mu + |\mathbf{b}|)^{\frac{p-2}{2}}\} (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{\frac{p-2}{2}}$, we derive by (2.10) that

$$\begin{aligned} |\mathbf{A}(\mathbf{b}) - \mathbf{A}(\mathbf{a})| &\leq (p-1)|\mathbf{b} - \mathbf{a}| \int_0^1 (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{2\frac{p-2}{2}}\, dt \\ &\leq C(p) M_{(|\mathbf{a}|, |\mathbf{b}|)} \int_0^1 |\mathbf{b} - \mathbf{a}| (\mu + |\mathbf{a} + t(\mathbf{b} - \mathbf{a})|)^{\frac{p-2}{2}}\, dt \\ &\leq C(p) M_{(|\mathbf{a}|, |\mathbf{b}|)} |\mathbf{F}(\mathbf{b}) - \mathbf{F}(\mathbf{a})|, \end{aligned} \tag{2.13}$$

where $M_{(|\mathbf{a}|, |\mathbf{b}|)} = 2 \max\{(\mu + |\mathbf{a}|)^{\frac{p-2}{2}}, (\mu + |\mathbf{b}|)^{\frac{p-2}{2}}\}$.

3. The Numerical Scheme

3.1. Preliminaries - DG notation

Let $T_h = \{E_i\}_{i=1}^{N_E}$ be a regular subdivision of Ω in triangular elements (without hanging nodes) with diameter h_{E_i}, where for simplicity we assume $h := \min_{E_i \in T_h} h_{E_i} = \max_{E_i \in T_h} h_{E_i}$. We denote by $\mathcal{E} = \mathcal{E}_I \bigcup \mathcal{E}_D$ all the edges, where $\mathcal{E}_I$ is the set of the interior edges of T_h, that is $\mathcal{E}_I = \{e : e = \partial E_{in} \bigcap \partial E_{out},\ \text{for } E_{in}, E_{out} \in T_h\}$ and $\mathcal{E}_D$ is the set of the *Dirichlet* boundary edges $\mathcal{E}_D = \{e : e = \partial E_{in} \bigcap \Gamma_D,\ E_{in}, \in T_h\}$. For each of $e \in \mathcal{E}_I$ we associate a unit normal vector $\mathbf{n}_e$. For $e \in \mathcal{E}_D$, $\mathbf{n}_e$ is considered to be the outward normal to $\partial\Omega$.

Define the following broken Sobolev spaces for $s \geq 2,\ p > 1$

$$W_h^{s,p}(T_h) := \{v \in L^p(\Omega) : v|_E \in W^{s,p}(E), \forall E \in T_h\}, \tag{3.1}$$

and the discontinuous finite element space $V_h^k(T_h) \subset W_h^{s,p}(T_h)$

$$V_h^k(T_h) := \{v \in L^p(\Omega) : v|_E \in \mathbb{P}^k(E), \forall E \in T_h\}, \tag{3.2}$$

where $\mathbb{P}^k(E)$ is the space of polynomials of degree less than or equal to k.

Let $e \in \mathcal{E}_I$, we define the average and the jump of $v \in W_h^{s,p}(T_h)$ on e by

$$\{v\} = \frac{1}{2}(v|_{E_{in}} + v|_{E_{out}}), \qquad \text{and} \qquad [v] = v|_{E_{in}} - v|_{E_{out}}. \tag{3.3}$$

In case of $e \in \mathcal{E}_D$, we define

$$\begin{aligned} \{v\} &= v|_{E_{in}}, \qquad \text{and} \qquad [v] = v|_{E_{in}}, \\ \{v\}_D &= \frac{1}{2}(v|_{E_{in}} + u_D), \qquad \text{and} \qquad [v]_D = v|_{E_{in}} - u_D. \end{aligned} \tag{3.4}$$

The space $W_h^{s,p}(T_h)$ is equipped with the broken DG norm, [27],[18],

$$\begin{aligned} \|\phi\|_{DG,p}^p = \sum_{E \in T_h} \int_E |\nabla \phi|^p \, dx + \sum_{e \in \mathcal{E}_I} \sigma h \int_e \Big|\frac{[\phi]}{h}\Big|^p ds + \\ \sum_{e \in \mathcal{E}_D} \sigma h \int_e \Big|\frac{[\phi]_D}{h}\Big|^p ds, \end{aligned} \tag{3.5}$$

where $p > 1$ and $\sigma > 0$ is a parameter.

3.2. Auxiliary results

Next, we summarize some results from the literature, which are going to be frequently used.

lemma 3.1. *(Trace inequalities). For $v_h \in V_h^k(T_h)$, and $v \in W_h^{s,p}(T_h)$ with $p > 1$ there exist positive constants $C_1(k,p), C_2(k,p), C_3(k,p)$ independent of the mesh size, such that* *(i)* $\|h^{\frac{1}{p}} v_h\|^p_{L^p(\Gamma_D)} \leq C_1 \sum_{E\in T_h} h\|v_h\|^p_{L^p(\partial E)}$

(ii) $\|v_h\|^p_{L^p(\partial E)} \leq C_2 h^{-1}\|v_h\|^p_{L^p(E)}$,

(iii) $\|v\|_{L^2(\partial E)} \leq C_3 h^{\frac{-1}{2}}(\|v\|_{L^2(E)} + h\|\nabla v\|_{L^2(E)})$.

Proof. The proofs of the above inequalities can be found in [7]. □

The Hölder, Young and Poincare's inequalities: let $1 < p, p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, and $\epsilon > 0$, then for $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ we have

$$\int_\Omega |uv|\,dx \leq \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}, \tag{3.6a}$$

$$\int_\Omega |uv|\,dx \leq \frac{\epsilon}{p}\|u\|^p_{L^p(\Omega)} + \frac{\epsilon^{\frac{-p'}{p}}}{p'}\|v\|^{p'}_{L^{p'}(\Omega)}. \tag{3.6b}$$

The generalized Poincare-Friedrichs inequality for $v \in W^{1,2}_h(T_h)$, see [28],

$$\|v\|_{L^2(\Omega)} \leq C\Big(\sum_{E\in T_h}\|\nabla v\|^2_{L^2(E)} + \sum_{e\in\mathcal{E}_I\bigcup\mathcal{E}_D}\frac{1}{h}\|[v]\|^2_{L^2(e)}\Big)^{\frac{1}{2}}. \tag{3.7}$$

3.3. The IPDG discretezation

For the simplification of the formulas below, we will often use $\int_\Omega u\,dx$ instead of $\sum_E\int_E u\,dx$. Inspired by the IPDG method in [12], we present the IPDG numerical scheme for discretizing the problem (2.1). We introduce the semi-linear form $B : W^{s,p}_h(T_h) \times W^{s,p}_h(T_h) \to \mathbb{R}$, such that for $u, \phi \in W^{s,p}_h(T_h)$

$$\begin{aligned}
B(u,\phi) = &\sum_{E\in T}\int_E a(\nabla u)\nabla u\nabla\phi\,dx - \sum_{e\in\mathcal{E}_I}\int_e\{a(\nabla u)\nabla u\cdot\mathbf{n}_e\}[\phi]\,ds\\
&-\sum_{e\in\mathcal{E}_I}\int_e\{a\Big(\frac{[u]}{h}\Big)\nabla\phi\cdot\mathbf{n}_e\}[u]\,ds + \sum_{e\in\mathcal{E}_I}\frac{\sigma}{h}\int_e a\Big(\frac{[u]}{h}\Big)[u][\phi]\,ds +\\
&-\sum_{e\in\mathcal{E}_D}\int_e a\Big(\frac{[u]_D}{h}\Big)\nabla\phi\cdot\mathbf{n}_e[u]_D\,ds + \sum_{e\in\mathcal{E}_D}\frac{\sigma}{h}\int_e a\Big(\frac{[u]_D}{h}\Big)[u]_D\phi\,ds\\
&-\sum_{e\in\mathcal{E}_D}\int_e a(\nabla u)\nabla u\cdot\mathbf{n}_e\phi\,ds,
\end{aligned} \tag{3.8}$$

where $\sigma := \sigma(k,p)$ is a positive parameter and will be specified in the error analysis. Giving an interpretation of the terms that appear in (3.8), we can say that, the second integral in (3.8) gives an approximation of the trace of the nonlinear flux and ensures the consistency of the method. The third integral, "symmetrizes" the flux form of $B(.,.)$ which is important for the numerical computations. The fourth integral penalizes the jumps on the interfaces and helps for achieving the discrete coercivity of $B(.,.)$. The rest terms defined on the boundary edges have similar meaning with the formers.

We also define the linear form

$$L(\phi) = \sum_{E \in T_h} \int_E f \phi dx. \tag{3.9}$$

The semi-dicrete problem is formulated as follows:
find $u_h \in W^{1,2}(0,T; V_h^k(T_h))$ such that

$$\begin{aligned} \int_\Omega \frac{\partial u_h(t)}{\partial t} \phi dx + B(u_h, \phi) =& L(\phi), \ \forall \phi \in V_h^k(T_h) \\ u_h(0) =& u_{0,DG}, \end{aligned} \tag{3.10}$$

where $u_{0,DG}$ is the approximation of the initial condition to the $V_h^k(T_h)$ space. Due to the assumed regularity (2.3) for the weak solution u (note that the jumps $[u] = 0$ on the interfaces), it is easy to show that u satisfies the variational formulation (3.10),

$$\int_\Omega \frac{\partial u(t)}{\partial t} \phi dx + B(u, \phi) = L(\phi), \ \forall \phi \in V_h^k(T_h). \tag{3.11}$$

For every $E \in T_h$, the DG solution of (3.10) is expressed as $u_h = \sum_i U_i^E(t) P_i(x)$ where U_i^E are the degrees of freedom and $P_i(x) \in \mathbb{P}^k(E)$ are the local polynomial basis functions. When this expression is substituted into (3.10), we obtain the following nonlinear ODE problem of finding the vector $\mathbf{U} = [.., U_i^E, ...]$ such that

$$\begin{aligned} M \frac{d\mathbf{U}(t)}{dt} + \mathbf{B}(\mathbf{U}(t)) =& \mathbf{L}(t), \\ U(0) =& u_h(0) \end{aligned} \tag{3.12}$$

where M is the block-diagonal mass matrix and the entries of $\mathbf{B}$ and $\mathbf{L}$ are specified by (3.8) and (3.9) respectively.

4. Fully discrete formulation

We discretize (3.12) with respect to time using s-stage Diagonally Implicit Runge-Kutta methods (s-DIRK), [29]. Hereafter, we denote by Δt the time step and with $\mathbf{U}^n$ the approximation of $\mathbf{U}(t_n)$ at time $t_n = n\Delta t, \ n = 0, 1, 2, ...$ If $\tau_i, i = 1, .., s$ are the quadrature points, b_i are the weights and $a_{ij}, j = 1, ..., i$ are the entries of Bucher's table, the s-DIRK method for the problem (3.12) is given by

$$M \Delta \mathbf{U}^{n,i} = - \Delta t \sum_{j=1}^{i} a_{ij} \Big(\mathbf{B}(\mathbf{U}^{n,j}) - \mathbf{L}(t^{n,j}) \Big), \ i = 1(1)s \tag{4.1a}$$

$$M \mathbf{U}^{n+1} = M \mathbf{U}^n - \Delta t \sum_{i=1}^{s} b_i \Big(\mathbf{B}(\mathbf{U}^{n,i}) - \mathbf{L}(t^{n,i}) \Big), \tag{4.1b}$$

where $t^{n,i} = t^n + \tau_i \Delta t$ and $\Delta \mathbf{U}^{n,i} = \mathbf{U}^{n,i} - \mathbf{U}^n$. The computation of the intermediate solutions $\mathbf{U}^{n,i}$ in (4.1a) includes the solution of a nonlinear system, which is achieved by a Picard iterative process

$$\begin{aligned} &\text{for } l = 1, ..., l_M, \text{ compute } \mathbf{U}^{n,l} \text{ by} \\ \Big(M + a_{ii} \Delta t B_P(\mathbf{U}^{n,l-1})\Big)\mathbf{U}^{n,l} &= \mathbf{R}(\mathbf{U}^n, \mathbf{U}^j), \\ \text{set } \mathbf{U}^{n,i} &= \mathbf{U}^{n,l_M}, \end{aligned} \tag{4.2}$$

where $B_P(\mathbf{U}^{n,l-1})$ is the iterative matrix produced by the Picard linearization and $\mathbf{R}(\mathbf{U}^n, \mathbf{U}^j) := M\mathbf{U}^n - \Delta t \sum_{j=1}^{i-1} a_{ij} \Big(\mathbf{B}(\mathbf{U}^{n,j}) - \mathbf{L}(t^{n,j})\Big)$ is the residual computed using the previous solutions. In the present work, for computational efficiency, two low-storage variations of the Picard iterative process (4.2) are applied, (i) the element-Jacobi (PEJ) and (ii) the element Gauss-Seidel (PEGS). Both iterative approaches are simple applications of the Picard iterative method presented in [22]. In the element-Jacobi scheme, the full Picard matrix $B_P(\mathbf{U}^{n,l-1})$ is approximated only by the block diagonal entries, neglecting in that way the contribution of the off-diagonal matrix blocks, which arise through the evaluation of the numerical fluxes on the interfaces. The numerical fluxes are computed using the previous solution vector $\mathbf{U}^{n,l-1}$ and are added to the right-hand residual $\mathbf{R}$. The diagonal blocks of $B_P(\mathbf{U}^{n,l-1})$ represent small dense matrices and are associated with each element $E \in T_h$. The solution of the resulting PEJ system of (4.2) is performed element by element using LU factorization method. The convergence of the previous proposed PEJ iterative method can be further accelerated by using Gauss-Seidel strategy, giving in this way, the second mentioned PEGS iterative method. PEGS method applies the same splitting of the matrix $B_P(\mathbf{U}^{n,l-1})$, but follows a passing over the interfaces by computing the numerical fluxes using the latest available solution $\mathbf{U}^{n,l}$ or $\mathbf{U}^{n,l-1}$ where it is possible. In the numerical tests (see Section 7), the iterative process of (4.2) stops when $\|\mathbf{U}^{n,l} - \mathbf{U}^{n,l-1}\| < tol$ for a prescribed tolerance tol and then we set $\mathbf{U}^{n,i} := \mathbf{U}^{n,l}$. We point out that, PEGS method is expected to have similar convergence rates per Runge-Kutta cycle as the PEJ method, but more improved performance behavior in terms of CPU (in fact the stopping criterion $\|\mathbf{U}^{n,l} - \mathbf{U}^{n,l-1}\| < tol$ is achieved performing fewer iterations l than the PEJ method). Comparison between the two iterative methods will be shown in Section 7. Other higher-order iterative procedures (e.g. Newton) can be applied for computing the intermediate solutions of (4.1a). In many cases, the computation of the Jacobian matrix of $\mathbf{B}(\mathbf{U}(t))$ may increase the CPU time of the whole ODE solver, see examples in [22], and more advanced numerical techniques must be applied, see e.g. [30], [31]. Anyway, for the numerical tests presented in Section 7, the previous proposed Picard iterative methods have been found to be appropriate for solving (3.12).

5. A stability bound in $\|.\|_{DG,p}$ for the case of $\mu = 0$

In this section, we give a stability estimate (a priori bound) for the DG solution u_h in case of $\mu = 0$, (p-Laplace problem) and note that Gronwall's lemma is not used. Stability bounds can be also obtained working in different direction using the monotonicity properties of $B(.,.)$, which are presented later. Here, the stability bound uses the $\|.\|_{DG,p}$-norm (3.5).

lemma 5.1. *For the form (3.8) with $\mu = 0$, there are constants $\kappa > 0$, $C_D > 0$ such that*

$$B(\phi,\phi) \geq \kappa \|\phi\|^p_{DG,p} - \frac{C_D}{h^{p-1}} \|u_D\|^p_{\Gamma_D}, \ \forall \phi \in V_h^k(T_h). \tag{5.1}$$

Proof. Choosing $u_h = \phi$ in (3.8) we obtain

$$\begin{aligned} B(\phi,\phi) = & \sum_{E\in T_h} \int_E a(\nabla\phi)\nabla\phi\cdot\nabla\phi dx - \sum_{e\in\mathcal{E}_I}\int_e \{a(\nabla\phi)\nabla\phi\cdot\mathbf{n}_e\}[\phi] ds \\ & - \sum_{e\in\mathcal{E}_I}\int_e \{a\Big(\frac{[\phi]}{h}\Big)\nabla\phi\cdot\mathbf{n}_e\}[\phi]ds + \sum_{e\in\mathcal{E}_I}\frac{\sigma}{h}\int_e a\Big(\frac{[\phi]}{h}\Big)[\phi][\phi]ds \\ & - \sum_{e\in\mathcal{E}_D}\int_e a\Big(\frac{(\phi-u_D)}{h}\Big)\nabla\phi\cdot\mathbf{n}_e(\phi-u_D)ds \\ & + \sum_{e\in\mathcal{E}_D}\frac{\sigma}{h}\int_e a\Big(\frac{(\phi-u_D)}{h}\Big)(\phi-u_D)\phi ds \\ & - \sum_{e\in\mathcal{E}_D}\int_e a(\nabla\phi)\nabla\phi\cdot\mathbf{n}_e\phi ds = \\ & T_1 - T_2 - T_3 + T_4 - T_5 + T_6 - T_7 \end{aligned}$$

For the term T_1, we have

$$T_1 = \sum_{E\in T_h}\int_E a(\nabla\phi)\nabla\phi\cdot\nabla\phi\, dx = \sum_{E\in T_h}\int_E |\nabla\phi|^p\, dx.$$

For the rest terms, applying inequalities (3.6), Lemma 3.1 and introducing constants $C_{i,\varepsilon} := C_i(\varepsilon, p, p')$ whit $\varepsilon > 0$, it follows that

$$\begin{aligned} T_2 \leq & \Big|\sum_{e\in\mathcal{E}_I}\int_e \{a(\nabla\phi)\nabla\phi\cdot\mathbf{n}_e\}[\phi]\, ds\Big| \leq \sum_{e\in\mathcal{E}_I}\int_e \{\Big|a(\nabla\phi)\nabla\phi\Big|\}\Big|[\phi]\Big|\, ds \leq \\ & \sum_{e\in\mathcal{E}_I}\int_e h^{\frac{1}{p'}}\{\Big|\nabla\phi\Big|^{p-1}\}\frac{\Big|[\phi]\Big|}{h^{\frac{1}{p'}}}\, ds \leq \end{aligned}$$

$$\sum_{e\in\mathcal{E}_I}\Big(\int_e \Big(h^{\frac{1}{p'}}\{\Big|\nabla\phi\Big|^{\frac{p}{p'}}\}\Big)^{p'}ds\Big)^{1/p'}\Big(\int_e\Big(\frac{\Big|[\phi]\Big|}{h^{\frac{1}{p'}}}\Big)^p ds\Big)^{1/p}\le$$

$$\sum_{e\in\mathcal{E}_I}\Big(\int_e h\{\Big|\nabla\phi\Big|^{\frac{p}{p'}}\}^{p'}ds\Big)^{1/p'}\Big(\int_e h\Big|\frac{[\phi]}{h}\Big|^{p-2}\Big|\frac{[\phi]}{h}\Big|^2 ds\Big)^{1/p}\le$$

$$\sum_{e\in\mathcal{E}_I}\Big(C_{2,\varepsilon}\int_e h\Big|\nabla\phi\Big|^p ds+\frac{h}{C_{2,\varepsilon}}\int_e h\Big|\frac{[\phi]}{h}\Big|^{p-2}\Big|\frac{[\phi]}{h}\Big|^2 ds\Big)\le$$

$$3C_{2,\varepsilon}\sum_{E\in T_h}\int_E\Big|\nabla\phi\Big|^p dx+\frac{h}{C_{2,\varepsilon}}\sum_{e\in\mathcal{E}_I}\int_e\Big|\frac{[\phi]}{h}\Big|^p ds.$$

For T_3, working in the same way as for T_2 we have

$$T_3\le\sum_{e\in\mathcal{E}_I}\int_e a\Big(\frac{[\phi]}{h}\Big)\Big|\frac{[\phi]}{h}\Big|h^{\frac{1}{p'}+\frac{1}{p}}\{\Big|\nabla\phi\Big|\}\,ds\le\sum_{e\in\mathcal{E}_I}\int_e\Big|\frac{[\phi]}{h}\Big|^{\frac{p}{p'}}h^{\frac{1}{p'}}h^{\frac{1}{p}}\{\Big|\nabla\phi\Big|\}\,ds\le$$

$$\sum_{e\in\mathcal{E}_I}\Big(\int_e\Big|\frac{[\phi]}{h}\Big|^p h\,ds\Big)^{\frac{1}{p'}}\Big(\int_e\Big(h^{\frac{1}{p}}\{\Big|\nabla\phi\Big|\}\Big)^p ds\Big)^{1/p}\le$$

$$\sum_{e\in\mathcal{E}_I}\Big(\frac{1}{C_{3,\varepsilon}}\int_e\Big|\frac{[\phi]}{h}\Big|^p h\,ds+C_{3,\varepsilon}\int_e h\{\Big|\nabla\phi\Big|\}^p ds\Big)\le$$

$$\frac{h}{C_{3,\varepsilon}}\sum_{e\in\mathcal{E}_I}\int_e\Big|\frac{[\phi]}{h}\Big|^p ds+3C_{3,\varepsilon}\sum_{E\in T_h}\int_E\Big|\nabla\phi\Big|^p ds.$$

A straightforward computation for the term T_4 gives

$$T_4=\sum_{e\in\mathcal{E}_I}h\sigma\int_e\Big|\frac{[\phi]}{h}\Big|^p ds.$$

For the term T_5 applying the same steps as for T_3 yields

$$T_5\le\frac{h}{C_{5,\varepsilon}}\sum_{e\in\mathcal{E}_D}\int_e\frac{|\phi-u_D|^p}{h}ds+3C_{5,\varepsilon}\sum_{E_D\in T_h}\int_{E_D}\Big|\nabla\phi\Big|^p ds,$$

where $E_D\in T_h$ are the boundary elements: $\{E\in T_h:\partial E\cap\Gamma_D\neq\emptyset\}$.
Term T_6 can be bounded as follows

$$T_6=\sum_{e\in\mathcal{E}_D}\sigma h\int_e\frac{|\phi-u_D|}{h}^{p-2}\frac{(\phi-u_D)}{h}\frac{(\phi-u_D+u_D)}{h}ds=$$

$$\sum_{e\in\mathcal{E}_D}\sigma h\int_e\Big(\frac{|\phi-u_D|}{h}\Big)^p ds-\sum_{e\in\mathcal{E}_D}\sigma h\int_e\Big(\frac{|\phi-u_D|}{h}\Big)^{p-2}\frac{(\phi-u_D)(-u_D)}{h^2}ds\ge$$

$$\sum_{e\in\mathcal{E}_D}\sigma h\int_e\Big(\Big|\frac{\phi-u_D}{h}\Big|\Big)^p ds-\sum_{e\in\mathcal{E}_D}\sigma h\int_e\Big(\frac{|\phi-u_D|}{h}\Big)^{p-1}\frac{(-u_D)}{h}ds\ge$$

$$(1-C_{6,\varepsilon})\sum_{e\in\mathcal{E}_D}\sigma h\int_e\Big|\frac{\phi-u_D}{h}\Big|^p ds-\frac{1}{C_{6,\varepsilon}}\sum_{e\in\mathcal{E}_D}\sigma h\int_e\frac{|u_D|^p}{h^{p-1}}ds.$$

Similarly, adding $u_D - u_D$ the term T_7 can be bounded

$$T_7 \le \sum_{e\in\mathcal{E}_D} \int_e \Big|\nabla\phi\Big|^{p-2}\Big|\nabla\phi\Big|\Big|(\phi - u_D + u_D)\Big|\, ds \le C_{7,\varepsilon} \sum_{E_D\in T_h} \int_{E_D} \Big|\nabla\phi\Big|^p dx$$
$$+ \frac{1}{C_7,\varepsilon} \sum_{e\in\mathcal{E}_D} \sigma h \int_e \Big|\frac{\phi - u_D}{h}\Big|^p + \frac{|u_D|^p}{h^{p-1}}\, ds.$$

In the previous inequalities, choosing $C_{i,\varepsilon}$ such that

$$3C_{2,\varepsilon} + 3C_{3,\varepsilon} + 3C_{5,\varepsilon} + C_{7,\varepsilon} \le \frac{1}{2},$$

and choosing the parameter σ to satisfy the following relations

$$\sigma > 1,\ (1 - C_{6,\varepsilon})\sigma \ge \frac{1}{C_{5,\varepsilon}} + \frac{1}{C_{7,\varepsilon}},\ \sigma > \frac{1}{C_{2,\varepsilon}} + \frac{1}{C_{3,\varepsilon}},$$

while keeping $h \le 1$, we can find $\kappa > 0$ and C_D, in order (5.1) to be true. □

Now, choosing $\phi = u_h(t)$ in (3.10) and using (5.1), we have

$$\frac{d}{2\,dt}\|u_h\|^2_{L^2(\Omega)} + \kappa\|u_h(t)\|^p_{DG,p} \le |L(u_h(t))| + \frac{C_D}{h^{p-1}}\|u_D\|^p_{\Gamma_D}. \tag{5.2}$$

Applying inequality (3.6) on the right hand side of (5.2), we get

$$|L(u_h(t))| \le \frac{1}{C_{8,\varepsilon}}\|f(t)\|^{p'}_{L^{p'}(\Omega)} + C_{8,\varepsilon}\|u_h(t)\|^p_{L^p(\Omega)}. \tag{5.3}$$

Based on the discrete embeddings, see [27],

$$\|\phi\|^p_{L^p(\Omega)} \le C_p\Big(\sum_E \int_E |\nabla\phi|^p + \sum_{e\in\mathcal{E}} \frac{1}{h^{p-1}} \int_e \Big|[\phi]\Big|^p\Big),\ \forall\phi \in V_h^k(T_h), \tag{5.4}$$

we can easily show that $\|u_h\|^p_{L^p(\Omega)} \le C_p\|u_h\|^p_{DG,p} + \frac{C_p}{h^{p-1}}\|u_D\|^p_{\Gamma_D}$.

Thus, inserting (5.3) into (5.2) and then applying inequality (5.4), we obtain for $C_{8,\varepsilon} = \frac{\kappa}{2}$ that

$$\frac{d}{2\,dt}\|u_h\|^2_{L^2(\Omega)} \ +\ \frac{\kappa}{2}\|u_h\|^p_{DG,p} \quad\le\quad \frac{1}{C\kappa, p}\|f\|^{p'}_{L^{p'}(\Omega)} \ +\ \frac{C_D}{h^{p-1}}\|u_D\|^p_{\Gamma_D}. \tag{5.5}$$

Integrating from 0 to t, we get the following stability bound for u_h,

$$\|u_h(t)\|^2_{L^2(\Omega)} + \kappa\int_0^t \|u_h(\tau)\|^p_{DG,p}\, d\tau \le \|u_{0h}\|^2_{L^2(\Omega)}$$
$$+ C\int_0^t \|f(\tau)\|^{p'}_{L^{p'}(\Omega)} + \frac{C_D}{h^{p-1}}\|u_D(\tau)\|^p_{\Gamma_D}\, d\tau. \tag{5.6}$$

6. Continuous in time a priori error estimates

Next, we give an error estimate on how close is the IPDG solution u_h of (3.10) to u of (3.11), that is an estimate for

$$\|u-u_h\|^2_{\mathbf{F},DG} = \sum_E \|\mathbf{F}(\nabla u)-\mathbf{F}(\nabla u_h)\|^2_{L^2(E)}+\sum_{e\in\mathcal{E}_I} \sigma h\|\mathbf{F}\Big(\frac{[u]}{h}\Big)-\mathbf{F}\Big(\frac{[u_h]}{h}\Big)\|^2_{L^2(e)},+$$
$$\sum_{e\in\mathcal{E}_D} \sigma h\|\mathbf{F}\Big(\frac{[u]_D}{h}\Big)-\mathbf{F}\Big(\frac{[u_h]_D}{h}\Big)\|^2_{L^2(e)}, \quad (6.1)$$

where the function $\mathbf{F}$ has been defined in (2.4) and the jumps $[.]$ in (3.3) and (3.4). We mention that similar error formula has been used in [32], where a LDG method studied for (p,δ)-structure problems. We consider the case where the solution u has the regularity (2.3), $u_h \in V_h^1(T_h)$ and $\mathcal{I}u \in V_h^1(T_h)$ is the Scott-Zhang interpolant of u, [33]. For problem (2.1), we suppose that $u_D \in \mathbb{P}^1(\mathcal{E}_D)$ and the parameter μ is such that (for example $\mu = 1$)

$$\int_0^1 (\mu + |\mathbf{a}+s(\mathbf{b}-\mathbf{a})|)^{p-2}\, ds \geq 1, \quad (6.2)$$

where in (6.2), $\mathbf{a}$ represents u or u_h, either their gradients and $\mathbf{b}$ takes the role of $\mathcal{I}u$ or its gradient. In the error analysis, we will make use of the following approximation result, which has been proved in [20].

lemma 6.1. *Let $u \in W^{s\geq 2,p}(\Omega)$ with $\mathbf{F}(\nabla u) \in W^{1,2}(\Omega)$, and $\mathcal{I}u \in V_h^1(T_h)$ its Scott-Zhang interpolant. Then there are constants $C_1, C_2 > 0$ independent of h such that*

$$\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u)\|^2_{L^2(E)} \leq C_1 h^2 \|\nabla \mathbf{F}(\nabla u)\|^2_{L^2(S_E)}, \ \forall E \in T_h, \quad (6.3)$$

where S_E is a domain made of the neighboring elements of E in T_h.

Corollary 6.2. *Under the assumptions of Lemma 6.1 and (2.3) the following estimate holds true for $t > 0$*

$$\|u - \mathcal{I}u\|^2_{\mathbf{F},DG} < Ch^2 \sum_{E\in T_h} \|\nabla \mathbf{F}(\nabla u)\|^2_{L^2(E)}. \quad (6.4)$$

for $C > 0$ independent of h.

Proof. We observe for u and the Scott-Zhang interpolant $\mathcal{I}u$ that $[u] = [\mathcal{I}u] = 0$ on every $e \in \mathcal{E}$. The estimate (6.4) follows immediately by the definition (6.1) and the approximation result (6.3). □

Proposition 6.3. *Under the assumptions (6.2), we can obtain the following estimates*

$$\|u - \mathcal{I}u\|^2_{L^2(\Omega)} \leq C_{\Omega,p,u,\mathcal{I}u} \|u - \mathcal{I}u\|^2_{\mathbf{F},DG}, \quad (6.5a)$$
$$\|u_h - \mathcal{I}u\|^2_{L^2(\Omega)} \leq C \|u_h - \mathcal{I}u\|^2_{\mathbf{F},DG}. \quad (6.5b)$$

Proof. We recall inequality (3.7) for $v := u - \mathcal{I}u$ and consequently we apply (2.8b) and (2.13) to obtain

$$\|u - \mathcal{I}u\|^2_{L^2(\Omega)} \leq C_{\Omega,p,u}\|u - \mathcal{I}u\|^2_{\mathbf{A},DG} \leq C_{\Omega,p,u}\|u - \mathcal{I}u\|^2_{\mathbf{F},DG}.$$

For the estimate (6.5b), let $e \in \mathcal{E}_D$, then we have that $|u_h - \mathcal{I}u|\big|_e \leq |u_h - u_D|\big|_e + |u_D - \mathcal{I}u|\big|_e$. Therefore, using the inequality $\|u_h - \mathcal{I}u\|_{L^2(\Omega)} \leq \|u_h - \mathcal{I}u\|_{DG,2}$ (see (3.7)) and then applying (2.8b) and (2.13) for every term of $\|u_h - \mathcal{I}u\|_{DG,2}$, we get

$$\|u_h - \mathcal{I}u\|^2_{L^2(\Omega)} \leq \|u_h - \mathcal{I}u\|_{DG,2} \leq C_{\Omega,p,u_h}\|u_h - \mathcal{I}u\|^2_{\mathbf{F},DG}.$$

□

lemma 6.4. *Under the assumptions of Lemma 6.1, there exist a $C > 0$ independent of h such that*

$$\sum_{e\in\mathcal{E}} h\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u)\|^2_{L^2(e)} \leq Ch^2 \sum_{E\in T_h} \|\nabla \mathbf{F}(\nabla u)\|^2_{L^2(E)}. \tag{6.6}$$

Proof. Using $v := \mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u)$ in inequality (iii) of Lemma 3.1 and summing over all edges, we have that

$$\sum_{e\in\mathcal{E}} h\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u)\|^2_{L^2(e)} \leq 3C \sum_{E\in T_h} (\|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u)\|^2_{L^2(E)} +$$
$$h^2\|\nabla(\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u))\|^2_{L^2(E)}) \leq Ch^2 \sum_{E\in T_h} \|\nabla \mathbf{F}(\nabla u)\|_{L^2(E)}.$$

□

lemma 6.5. *Let $\mathcal{I}u \in V_h^1(T_h)$ be the interpolant of u as in (6.3) and let $\phi = u_h - \mathcal{I}u$. For every edge $e \in \mathcal{E}$ there are $C_{1,\varepsilon}, C_{2,\varepsilon} > 0$ such that*

$$\Big|\int_e \{a(\nabla u_h)\nabla u_h - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\} \cdot \mathbf{n}_e[\phi]\, ds\Big| \leq$$
$$C_{1,\varepsilon}\|\mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}u)\|^2_{L^2(E^{in}\bigcup E^{out})} + \frac{h}{C_{2,\varepsilon}}\Big\|\mathbf{F}\Big(\frac{[u_h]}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]}{h}\Big)\Big\|^2_{L^2(e)}. \tag{6.7}$$

Proof. Let $e = \partial E_{in} \bigcap \partial E_{out}$ (or $e \in \mathcal{E}_D$). Applying sequentially the inequalities (3.6) and Lemma 3.1 on the left hand side of (6.7), we have

$$\int_e h^{\frac{1}{2}}\Big|\{a(\nabla u_h)\nabla u_h - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\}\Big|\frac{1}{h^{\frac{1}{2}}}\Big|[\phi]\Big|\, ds \leq$$
$$\Big(\int_e h\Big|\{a(\nabla u_h)\nabla u_h - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\}\Big|^2 ds\Big)^{\frac{1}{2}}\Big(\int_e \frac{1}{h}\Big|[\phi]\Big|^2 ds\Big)^{\frac{1}{2}} \leq$$

$$\Big(\frac{h}{2}\|a(\nabla u_h)\nabla u_h - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\|^2_{L^2(e^{in})} + \frac{h}{2}\|...\|^2_{L^2(e^{out})}\Big)^{\frac{1}{2}}\Big(\frac{1}{h}\|[\phi]\|_{L^2(e)}\Big)^{\frac{1}{2}} \leq$$

$$C_{trc}\Big(\|a(\nabla u_h)\nabla u_h - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\|^2_{L^2(E_{in})} + \|...\|^2_{L^2(E_{out})}\Big)^{\frac{1}{2}}\Big(\frac{1}{h}\|[\phi]\|^2_{L^2(e)}\Big)^{\frac{1}{2}} \leq$$

$$C_{1,\varepsilon}\|\|a(\nabla u_h)\nabla u_h - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\|^2_{L^2(E_{in}\bigcup E_{out})} + \frac{h}{C_{2,\varepsilon}}\Big\|\frac{[\phi]}{h}\Big\|^2_{L^2(e)} \leq^{\Big(by\ (2.8b),(2.13)\Big)}$$

$$C_{1,\varepsilon}\|\mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}u)\|^2_{L^2(E_{in}\bigcup E_{out})} + \frac{h}{C_{2,\varepsilon}}\Big\|\mathbf{F}\Big(\frac{[u_h]}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]}{h}\Big)\Big\|^2_{L^2(e)},$$

where for simplicity, we used the notation: $L^2(e^{in}) := L^2(e \subset \partial E_{in})$. □

lemma 6.6. *Let $\mathcal{I}u \in V_h^1(T_h)$ be the interpolant as in (6.3) of the solution u and $\phi = u_h - \mathcal{I}u$. For every edge $e = \partial E^{in} \bigcap \partial E^{out}$ (or $e \in \mathcal{E}_D$) there are $C_{1,\varepsilon}, C_{2,\varepsilon} > 0$ such that*

$$\Big|\int_e \Big(a\Big(\frac{[u_h]}{h}\Big)[u_h] - a\Big(\frac{[\mathcal{I}u]}{h}\Big)[\mathcal{I}u]\Big)\{\nabla u_h - \nabla \mathcal{I}u\}\cdot \mathbf{n}_e\, ds\Big| \leq$$

$$\frac{h}{C_{2,\varepsilon}}\int_e \Big|\mathbf{F}\Big(\frac{[u_h]}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]}{h}\Big)\Big|^2 ds + C_{1,\varepsilon}\|\mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}u)\|^2_{L^2(E^{in}\bigcup E^{out})}. \quad (6.8)$$

Proof. Following the same steps as in proof of Lemma 6.5, by applying Hölder's inequality, trace inequality, consequently Young's inequality and (2.8b),(2.13), the relation (6.8) can be shown. □

Theorem 6.7. *Let u be the solution of (3.11) and let $\mathcal{I}u \in V_h^1(T_h)$ be its interpolant as in (6.3). Then for $\phi = u_h - \mathcal{I}u$ and $\varepsilon > 0$ there exist constants $C_{1,\varepsilon}, C_{2,\varepsilon}, C_{3,\varepsilon}$ such that the form B of (3.8) satisfies*

$$|B(u,\phi) - B(\mathcal{I}u,\phi)| \leq \frac{1}{C_{1,\varepsilon}}\|u - \mathcal{I}u\|^2_{\mathbf{F},DG} + C_{2,\varepsilon}\|\phi\|^2_{\mathbf{F},DG} + \quad (6.9)$$

$$\frac{2}{C_{3,\varepsilon}}\sum_{e\in\mathcal{E}}\int_e h|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u)|^2\, ds.$$

Proof. After a rearrangement of the terms of $B(u,\phi) - B(\mathcal{I}u,\phi)$, we have

$$\begin{aligned}\Big|B(u,\phi) - B(\mathcal{I}u,\phi)\Big| \le & \int_\Omega \Big|a(\nabla u)\nabla u - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\Big|\Big|\nabla\phi\Big|\,dx + \\ & \sum_{e\in\mathcal{E}_I}\int_e \{\Big|a(\nabla u)\nabla u - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\Big|\}\Big|[\phi]\Big|\,ds + \\ & \sum_{e\in\mathcal{E}_D}\int_e \Big|a(\nabla u)\nabla u - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\Big|\Big|\phi\Big|\,ds + \\ & \sum_{e\in\mathcal{E}_I}\int_e \Big|a\Big(\frac{[u]}{h}\Big)[u] - a\Big(\frac{[\mathcal{I}u]}{h}\Big)[\mathcal{I}u]\Big|\{\Big|\nabla\phi\Big|\}\,ds + \\ & \sum_{e\in\mathcal{E}_D}\int_e \Big|a\Big(\frac{[u]_D}{h}\Big)[u]_D - a\Big(\frac{[\mathcal{I}u]_D}{h}\Big)[\mathcal{I}u]_D\Big|\Big|\nabla\phi\Big|\,ds + \\ & \sum_{e\in\mathcal{E}_I}\sigma\int_e \Big(a\Big(\frac{[u]}{h}\Big)\frac{[u]}{h} - a\Big(\frac{[\mathcal{I}u]}{h}\Big)\frac{[\mathcal{I}u]}{h}\Big)\Big|[\phi]\Big|\,ds + \\ & \sum_{e\in\mathcal{E}_D}\sigma\int_e \Big(a\Big(\frac{[u]_D}{h}\Big)\frac{[u]_D}{h} - a\Big(\frac{[\mathcal{I}u]_D}{h}\Big)\frac{[\mathcal{I}u]_D}{h}\Big)\Big|\phi\Big|\,ds = \\ & T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7.\end{aligned}$$

The first term T_1 can be bounded by applying the Hölder-Young's inequalities (3.6) and consequently (2.8b),(2.13), as follows

$$\begin{aligned}T_1 \le & \Big(\sum_{E\in T_h}\int_E \Big|a(\nabla u)\nabla u - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\Big|^2 dx\Big)^{1/2}\Big(\sum_{E\in T_h}\int_E \Big|\nabla\phi\Big|^2 dx\Big)^{1/2} \le \\ & \frac{1}{C_{2,\varepsilon}}\int_\Omega \Big|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u)\Big|^2 dx + C_{1,\varepsilon}\int_\Omega \Big|\mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}u)\Big|^2.\end{aligned}$$

The term T_2 can be bounded by applying the same steps as in Lemma 6.5,

$$\begin{aligned}T_2 \le & \sum_{e\in\mathcal{E}_I}\int_e h^{\frac12}\Big|\{a(\nabla u)\nabla u - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\}\Big|\frac{1}{h^{\frac12}}\Big|[\phi]\Big|\,ds \le \\ & \sum_{e\in\mathcal{E}_I}\Big[\Big(\int_e h\Big|\{a(\nabla u)\nabla u - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\}\Big|^2 ds\Big)^{\frac12}\Big(\int_e \frac{1}{h}\Big|[\phi]\Big|^2 ds\Big)^{\frac12}\Big] \le \\ & \frac{1}{C_{2,\varepsilon}}\sum_{e\in\mathcal{E}_I}\int_e h\Big|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u)\Big|^2 ds + C_{1,\varepsilon}h\Big\|\mathbf{F}\Big(\frac{[u_h]}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]}{h}\Big)\Big\|^2_{L^2(e)}.\end{aligned}$$

Analogously, for the term T_3, we obtain that

$$\begin{aligned}T_3 \le & \sum_{e\in\mathcal{E}_D}\int_e h^{\frac12}\Big|a(\nabla u)\nabla u - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\Big|\frac{1}{h^{\frac12}}\Big|\phi\Big|\,ds \le \\ & \frac{1}{C_{2,\varepsilon}}\sum_{e\in\mathcal{E}_D}\int_e h\Big|\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}u)\Big|^2 ds + C_{1,\varepsilon}h\Big\|\mathbf{F}\Big(\frac{[u_h]_D}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]_D}{h}\Big)\Big\|^2_{L^2(e)}.\end{aligned}$$

The term T_4 can be bounded working in the same way as in Lemma 6.5,

$$T_4 \leq \sum_{e\in\mathcal{E}_I} \int_e h^{\frac{1}{2}} \Big| a\Big(\frac{[u]}{h}\Big)\frac{[u]}{h} - a\Big(\frac{[\mathcal{I}u]}{h}\Big)\frac{[\mathcal{I}u]}{h} \Big| h^{\frac{1}{2}} \{ \Big|\nabla\phi\Big| \}\, ds \leq$$

$$\sum_{e\in\mathcal{E}_I} \Big(\int_e h \Big| a\Big(\frac{[u]}{h}\Big)\frac{[u]}{h} - a\Big(\frac{[\mathcal{I}u]}{h}\Big)\frac{[\mathcal{I}u]}{h} \Big|^2 ds \Big)^{\frac{1}{2}} \Big(\int_e h \{ \Big|\nabla\phi\Big| \}^2 ds \Big)^{\frac{1}{2}} \leq$$

$$\frac{1}{C_{2,\varepsilon}} \sum_{e\in\mathcal{E}_I} \Big(\int_e h \Big| \mathbf{F}\Big(\frac{[u]}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]}{h}\Big) \Big|^2 ds + 3C_{1,\varepsilon} \sum_{E\in T_h} \Big(\int_E \Big| \mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}u) \Big|^2 dx.$$

Applying the same steps for T_5, we get

$$T_5 \leq \frac{1}{C_{2,\varepsilon}} \sum_{e\in\mathcal{E}_D} \Big(\int_e h \Big| \mathbf{F}\Big(\frac{[u]_D}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]_D}{h}\Big) \Big|^2 ds +$$

$$3C_{1,\varepsilon} \sum_{E_D\in T_h} \Big(\int_{E_D} \Big| \mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}u) \Big|^2 dx.$$

The last terms T_6 and T_7 are bounded by applying the same steps as before

$$T_6 \leq \sigma \sum_{e\in\mathcal{E}_I} \int_e \Big| a\Big(\frac{[u]}{h}\Big)\frac{[u]}{h} - a\Big(\frac{[\mathcal{I}u]}{h}\Big)\frac{[\mathcal{I}u]}{h} \Big| h^{\frac{1}{2}} \Big| \frac{[\phi]}{h^{\frac{1}{2}}} \Big| ds \leq$$

$$\sigma \sum_{e\in\mathcal{E}_I} \Big(\int_e h \Big| a\Big(\frac{[u]}{h}\Big)\frac{[u]}{h} - a\Big(\frac{[\mathcal{I}u]}{h}\Big)\frac{[\mathcal{I}u]}{h} \Big|^2 \Big)^{\frac{1}{2}} \Big(\int_e \frac{1}{h} \Big|[\phi]\Big| ds \Big)^{\frac{1}{2}} \leq$$

$$\frac{1}{C_{2,\varepsilon}} \sum_{e\in\mathcal{E}_I} \int_e h \Big| \mathbf{F}\Big(\frac{[u]}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]}{h}\Big) \Big|^2 ds + C_{1,\varepsilon} \sum_{e\in\mathcal{E}_I} \int_e h \Big| \mathbf{F}\Big(\frac{[u_h]}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]}{h}\Big) \Big|^2 ds.$$

$$T_7 \leq \frac{1}{C_{2,\varepsilon}} \sum_{e\in\mathcal{E}_D} \int_e h \Big| \mathbf{F}\Big(\frac{[u]_D}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]_D}{h}\Big) \Big|^2 ds +$$

$$C_{1,\varepsilon} \sum_{e\in\mathcal{E}_D} \int_e h \Big| \mathbf{F}\Big(\frac{[u_h]_D}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]_D}{h}\Big) \Big|^2 ds.$$

Choosing appropriate the constants $C_{i,\varepsilon}$ above and by gathering the bounds together, we can derive (6.9). □

Theorem 6.8. *Let $\mathcal{I}u \in V_h^1(T_h)$ be the interpolant of the solution u. The form $B(.,.)$ is monotone with respect to second argument, in the sense that there is a $\kappa_0 > 0$ such that*

$$B(u_h, u_h - \mathcal{I}u) - B(\mathcal{I}u, u_h - \mathcal{I}u) > \kappa_0 \|u_h - \mathcal{I}u\|^2_{\mathbf{F},DG}. \tag{6.10}$$

Proof. Denoting $\phi = u_h - \mathcal{I}u$ and after rearranging the terms, we obtain that

$$\begin{aligned}
B(u_h, u_h - \mathcal{I}u) - B(\mathcal{I}u, u_h - \mathcal{I}u) = \\
\sum_{E\in T_h} \int_E \Big(a(\nabla u_h)\nabla u_h - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u \Big) \nabla\phi \, dx - \\
\sum_{e\in\mathcal{E}_I} \int_e \{a(\nabla u_h)\nabla u_h - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u\}[\phi] \, ds - \\
\sum_{e\in\mathcal{E}_D} \int_e \Big(a(\nabla u_h)\nabla u_h - a(\nabla \mathcal{I}u)\nabla \mathcal{I}u \Big) \phi \, ds - \\
\sum_{e\in\mathcal{E}_I} \int_e \Big(a\Big(\frac{[u_h]}{h}\Big)[u_h] - a\Big(\frac{[\mathcal{I}u]}{h}\Big)[\mathcal{I}u] \Big) \{\nabla\phi\} \, ds - \\
\sum_{e\in\mathcal{E}_D} \int_e \Big(a\Big(\frac{[u_h]_D}{h}\Big)[u_h]_D - a(\frac{[\mathcal{I}u]_D}{h})[\mathcal{I}u]_D \Big) \nabla\phi \, ds \\
+ \sum_{e\in\mathcal{E}_I} \sigma \int_e \Big(a\Big(\frac{[u_h]}{h}\Big)\frac{[u_h]}{h} - a\Big(\frac{[\mathcal{I}u]}{h}\Big)\frac{[\mathcal{I}u]}{h} \Big) [\phi] \, ds + \\
\sum_{e\in\mathcal{E}_D} \sigma \int_e \Big(a\Big(\frac{[u_h]_D}{h}\Big)\frac{[u_h]_D}{h} - a\Big(\frac{[\mathcal{I}u]_D}{h}\Big)\frac{[\mathcal{I}u]_D}{h} \Big) \phi \, ds.
\end{aligned}$$

Using (2.12a), Lemma 6.5 and Lemma 6.6, we have

$$\begin{aligned}
B(u_h, u_h - \mathcal{I}u) - B(\mathcal{I}u, u_h - \mathcal{I}u) \geq C_p \sum_{E\in T_h} \int_E |\mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}u)|^2 \, dx - \\
C_{1,\varepsilon} \sum_{E\in T_h} \|\mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}u)\|^2_{L^2(E)} - \frac{h}{C_{2,\varepsilon}} \sum_{e\in\mathcal{E}_I} \|\mathbf{F}\Big(\frac{[u_h]}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]}{h}\Big)\|^2_{L^2(e)} - \\
\frac{h}{C_{2,\varepsilon}} \sum_{e\in\mathcal{E}_D} \|\mathbf{F}\Big(\frac{[u_h]_D}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u)]_D}{h}\Big)\|^2_{L^2(e)} - C_{1,\varepsilon} \sum_{E\in T_h} \|\mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}u)\|^2_{L^2(E)} + \\
\sigma \sum_{e\in\mathcal{E}_I} h \|\mathbf{F}\Big(\frac{[u_h]}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]}{h}\Big)\|^2_{L^2(e)} + \sigma \sum_{e\in\mathcal{E}_D} h \|\mathbf{F}\Big(\frac{[u_h]_D}{h}\Big) - \mathbf{F}\Big(\frac{[\mathcal{I}u]_D}{h}\Big)\|^2_{L^2(e)}.
\end{aligned}$$

Gathering the bounds and choosing appropriately the constants $C_{i,\varepsilon}$ and σ, for example $2C_{1,\varepsilon} < \frac{C_p}{2}$ and $\sigma - \frac{1}{C_{2,\varepsilon}} > \frac{1}{2}$, we can find κ_0 such that the relation (6.10) to be true. □

Next, we give the estimate for the approximation error $u_h - u$.

Theorem 6.9. *Under the assumptions (2.3) and (6.2), and choosing* $u_h(0) :=$

$\mathcal{I}u_0$, there exist constants κ_0 and $C > 0$ such that: for $t \in (0, T]$

$$\|u(t) - u_h(t)\|^2_{L^2(\Omega)} + \frac{\kappa_0}{2}\int_0^t \|u(\tau) - u_h(\tau)\|^2_{\mathbf{F},DG}\, d\tau \leq \|u(t) - \mathcal{I}u(t)\|^2_{L^2(\Omega)} + C\int_0^t \|\partial_t\big(u(\tau) - \mathcal{I}u(\tau)\big)\|^2_{L^2(\Omega)}\, d\tau + Ch^2\int_0^t \|\nabla\mathbf{F}(\nabla u(\tau))\|_{L^2(\Omega)}\, d\tau. \quad (6.11)$$

Proof. We have by variational formulations (3.10) and (3.11) for $t > 0$ that

$$\int_\Omega \partial_t u_h \phi\, dx \;+\; B(u_h, \phi) \;=\; \int_\Omega \partial_t u \phi\, dx \;+\; B(u, \phi),\ \forall \phi \;\in\; V_h^k(T_h). \quad (6.12)$$

Setting $\phi = u_h - \mathcal{I}u$ and adding $-\int_\Omega \partial_t \mathcal{I}u\phi\, dx - B(\mathcal{I}u, \phi)$ on both sides of (6.12), we get

$$\int_\Omega \partial_t(\phi)\phi\, dx + B(u_h, \phi) - B(\mathcal{I}u, \phi) = \int_\Omega \partial_t(u - \mathcal{I}u)\phi\, dx + B(u, \phi) - B(\mathcal{I}u, \phi). \quad (6.13)$$

Now, we use (6.6) in (6.9) and then we use the derived result on the right hand side of (6.13). Consequently, we make use of (6.10) to the left hand side of (6.13), we eventually end up with the following relation

$$\frac{1}{2}\frac{\partial}{\partial t}\|\phi\|^2_{L^2(\Omega)} + \kappa_0\|\phi\|^2_{\mathbf{F},DG} \leq \int_\Omega \partial_t(u - \mathcal{I}u)\phi\, dx + \frac{1}{C_{1,\varepsilon}}\|u - \mathcal{I}u\|^2_{\mathbf{F},DG} + C_{2,\varepsilon}\|\phi\|^2_{\mathbf{F},DG} + C_{3,\varepsilon}h^2\|\nabla\mathbf{F}(\nabla u)\|_{L^2(\Omega)}. \quad (6.14)$$

Applying (3.6) on the first term of the right hand side of (6.14) and then using discrete embeddings (6.5b) yields

$$\frac{1}{2}\frac{\partial}{\partial t}\|\phi\|^2_{L^2(\Omega)} + \kappa_0\|\phi\|^2_{\mathbf{F},DG} \leq \frac{1}{4\kappa_0}\|\partial_t(u - \mathcal{I}u)\|^2_{L^2(\Omega)} + \frac{\kappa_0}{4}\|\phi\|^2_{\mathbf{F},DG}. \frac{1}{C_{1,\varepsilon}}\|u - \mathcal{I}u\|^2_{\mathbf{F},DG} + C_{2,\varepsilon}\|\phi\|^2_{\mathbf{F},DG} + C_{3,\varepsilon}h^2\|\nabla\mathbf{F}(\nabla u)\|_{L^2(\Omega)}. \quad (6.15)$$

Next, using (6.4) and choosing $C_{2,\varepsilon} = \frac{\kappa_0}{4}$ into (6.15), we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_h - \mathcal{I}u\|^2_{L^2(\Omega)} + \frac{\kappa_0}{2}\|u_h - \mathcal{I}u\|^2_{\mathbf{F},DG} \leq \frac{1}{4\kappa_0}\|\partial_t(u - \mathcal{I}u)\|^2_{L^2(\Omega)} + C_\varepsilon h^2\|\nabla\mathbf{F}(\nabla u)\|_{L^2(\Omega)}. \quad (6.16)$$

We integrate (6.16) from 0 to t:

$$\|u_h(t) - \mathcal{I}u(t)\|^2_{L^2(\Omega)} + \kappa_0\int_0^t \|u_h(\tau) - \mathcal{I}u(\tau)\|^2_{\mathbf{F},DG}\, d\tau \leq \|u_{0,h} - \mathcal{I}u_0\|^2_{L^2(\Omega)} + \int_0^t \Big(\frac{1}{2\kappa_0}\|\partial_t(u(\tau) - \mathcal{I}u(\tau))\|^2_{L^2(\Omega)} + C_\varepsilon h^2\|\nabla\mathbf{F}(\nabla u(\tau))\|_{L^2(\Omega)}\Big)d\tau. \quad (6.17)$$

Observing that $u_h(0) - \mathcal{I}u_0 = 0$ and applying the triangle inequality

$$\|u_h(t) - u(t)\|_* \leq \|u_h(t) - \mathcal{I}u(t)\|_* + \|u(t) - \mathcal{I}u(t)\|_*,$$

in (6.17), we can deduce the estimate (6.11). □

Using further the estimates (6.5a) in (6.11), we prove the following corollary.

Corollary 6.10. *Under the assumptions of Theorem 6.9, there is a* $C := C(\|\nabla\mathbf{F}(\nabla u(t))\|^2_{L^2(0,T;L^2(\Omega))}, \|\nabla\mathbf{F}(\nabla u(t))\|^2_{L^2(\Omega)}\|\nabla u_t\|^2_{L^2(0,T;L^2(\Omega))})$ *such that*

$$\int_0^t \|u(\tau) - u_h(\tau)\|^2_{\mathbf{F},DG}\, d\tau \leq Ch^2, \tag{6.18}$$

Proof. The assertion follows by the application of the interpolation estimate of Lemma 6.1 and the estimate (6.5a) on the terms of the right hand side of (6.11). □

7. Numerical examples

In this section, we present numerical results to illustrate the performance of the proposed IPDG method for solving problem (2.1) and to verify the theoretical results of the previous section. The numerical examples have been performed for $p = 2.3$, $p = 2.5$, $p = 3$, using $\mu = 1$, $\sigma = 2.5$ (see (3.8)). The Picard iterative procedure was stopped until the tolerance value satisfied by $tol \leq 1.E-07$.

The domain is $\Omega := [-2, 2] \times [-2, 2]$, where $\Gamma_D = \partial\Omega$ and the data f, u_D of (2.1) are specified so that the exact solution is

$$u(x, y, t) = B(t)\sin(x + y), \tag{7.1}$$

where $B(t) = 1 + exp(-100t)$. The initial unstructured mesh T_{h_0} is generated by a triangular mesh generator with $h_0 = 1$ and the next finer meshes T_{h_i} are obtained by subdividing the triangles to four equal triangles, $h_{i+1} = \frac{h_i}{2}$. The problem has been solved up to final time $T = 0.5$ using a second order, 1-stage DIRK method, [29]. In Figure 1 left, the T_{h_2} mesh of the domain Ω is presented and in Fig. 1 right, we plot the u_h solution computed on T_{h_2} mesh for the $p = 2.3$ test case.

In the first numerical test, the CPU time of the iterative methods PEJ and PEGS is compared. In Table 1, the CPU time for the $p = 2.3$ test case is given. As it was expected, for the same value of tol, PEGS performs faster and appears to be more efficient than the PEJ iterative method.

Next, we give examples for the convergence rate of the error,

$$e_{h_i} = \int_0^t \|u(\tau) - u_{h_i}(\tau)\|^2_{\mathbf{F},DG}\, d\tau, \tag{7.2}$$

where u_{h_i} is the IPDG solution and u is the solution (7.1). All the numerical tests have been performed using the PEGS method with $\Delta t < (\frac{h_i}{10})^p$. The numerical convergence rates r are computed by the formula $r = \frac{\ln(e_{h_i}/e_{h_{i+1}})}{\ln(2)}$. The results are shown in Table 2. We can observe that for all p-test cases the error (7.2) converges with the rate that has been predicted in the Corollary 6.10.

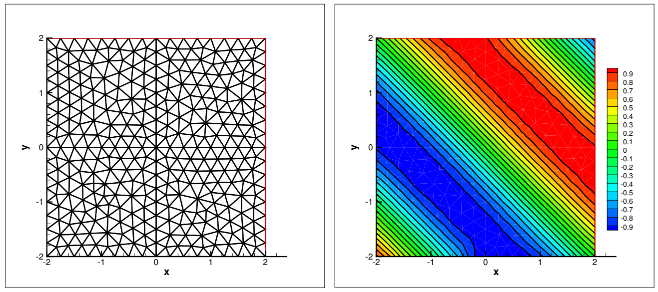

Figure 1: Left: The domain Ω with the T_{h_2} mesh. Right: The contours of u_h computed for the $p = 2.3$ test.

-	PEJ	PEGS
T_{h_i}	CPU for p=2.3	
$i = 0$	21.6263	13.5540
$i = 1$	73.4690	56.5072
$i = 2$	195.4758	186.6697
$i = 3$	712.597	625.4143

Table 1: CPUs for the two Picard iterative methods

-	p=2.3	p=2.5	p=3
T_{h_i}	rates r		
$i = 0$	-	-	-
$i = 1$	2.12	2.30	2.02
$i = 2$	2.10	2.06	2.05
$i = 3$	2.04	2.02	2.02

Table 2: Convergence rates for the three p-test cases.

8. Conclusions

In this work, an IPDG method was presented for approximating the solution of a quasilenar parabolic problem formulated in L^p-setting. The resulting nonlinear ODE system was discretized in time by s-DIRK methods applying two low-storage Picard iterative schemes for solving the resulting nonlinear systems. A stability bound were shown in the broken $\|.\|_{DG,p}$-norm for the IPDG solution. Optimal error estimates for the IPDG method were proved in the broken $\|.\|_{\mathbf{F},DG}$-norm for the case of $p > 2$. The theoretical results were validated by numerical tests for several values of $p > 2$.

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